

On Brown-Peterson cohomology of QX

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Introduction

Given a spectrum X and a generalized cohomology theory h with $h_*(X)$ known, what can we say about $h_*(\underline{X}_i)$ where \underline{X}_i denotes the i^{th} infinite loop space associated to X ? An obvious place to start investigating this question would be the case when X is a suspension spectrum $\Sigma^\infty Y$ where Y is a based space. In this case one has

$$\underline{X}_i \cong Q\Sigma^i Y \cong \operatorname{colim}_s \Sigma^{i+s} \Omega^s Y.$$

The mod p ordinary homology of such a space was computed by Kudo and Araki in the case $p = 2$ [2] and by Dyer and Lashof in the case p is odd [9]. Later, J. P. May determined the Bockstein spectral sequence in terms of that for Y . Since the rational homology of such a space is easy to determine, this gives us complete knowledge of ordinary homology of spaces of the form QX . As one might expect, the first extraordinary homology that was studied was the mod p K -theory. The first result is due to Hodgkin, dating back to the 1970's.

THEOREM 0.1 ([10]). $K_*(QS^0; Z/p) \cong Z/p[\iota, Q\iota, Q^2\iota, \dots][\iota^{-1}]$.

Here Q^i denotes the i^{th} iteration of Q , which is an analogue of the classical Dyer-Lashof-Kudo-Araki operation, defined up to a certain indeterminacy. Hodgkin had shown earlier in [11] that the indeterminacy was inevitable. Later in the beginning of the 1980's, Miller and Snaith determined [31] $K_*(QS^n; Z/2)$ as well as $K_*(QRP^n; Z/2)$. Finally, McClure succeeded in constructing a well-defined operation from $K_*(Y, Z/p^r)$ to $K_*(Y, Z/p^{r-1})$ ($r \geq 2$), for infinite loop spaces Y and described $K_*(QX; Z/p^r)$ ($r \geq 1$) in terms of the Bockstein spectral sequence for $K_*(X)$ ([30]) (which is equivalent to the knowledge of $K_*(X, Z/p^r)$ for all $r \geq 1$). The answer is too complicated to quote here, but

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we note that for an odd prime p , $K_*(QS^n; \mathbb{Z}/p)$ is a free commutative (in the graded sense) algebra with generators $\iota, Q\iota, Q^2\iota, \dots$ where ι is the “fundamental class”, that is, the image of the generator of $K_*(S^n)$ by the map induced by the map $S^n \rightarrow QS^n$ with Q as above.

Naturally the next cases to study would be the higher Morava K -theories. The computation by Hodgkin depended on the theorem of Atiyah that identifies the K -theory of the classifying space of a finite group with the completion of its representation ring at the augmentation ideal $K^*(BG) \cong R(G)^\wedge$ and the intimate relationship between the classifying spaces of the symmetric groups and QS^0 . Unfortunately, in the case of Morava K -theories, we only know that the dimension of $K(n)^*(BG)$ is equal to the number of n -tuples of commuting elements of order a power of p [14], provided that it is concentrated in even degrees (which is the case for symmetric groups ([14], [15])) and we do not have their functorial description. However, when $n = 1$, it turned out that these formulae were sufficient to recover the results of [10] (see [20]). Furthermore, by doing quite involved calculations of characteristic classes, the author computed $K(2)_*(QS^r)$ ([22]). The result is that one can define the operations Q_1, \dots, Q_{p+1} with indeterminacies, which are subject only to the relations of the form $Q_i Q_1 = 0$ if $1 < i \leq p$, $Q_{p+1} Q_1 = Q_1 Q_2$, which have to be interpreted suitably, and $K(2)_*(QS^r)$ is a free commutative algebra generated by iteration of these operations acting on the fundamental class.

However, the generalization of the method used in [22] seems to be difficult, as there is a consensus among the experts that in the case for the n^{th} Morava K -theory, among the generalized Dyer-Lashof operations there should be relations of length n , i.e., those of the form $Q_{i_1} \cdots Q_{i_n} = \sum a_{i,j} Q_{j_1} \cdots Q_{j_n}$, that are not generated by shorter relations. Since in the case for $n = 2$, the relations were derived from the computation of the image of $K(2)^*(B\Sigma_{p^2})$ in $K(2)^*(B\Sigma_p \wr \Sigma_p)$, a direct generalization of the method in [22] would involve n -factors computing the image of $K(n)^*(B\Sigma_{p^n})$ in $K(n)^*(\overbrace{B\Sigma_p \wr \cdots \wr \Sigma_p}^{n \text{ -factors}})$, which seems to be hopeless.

One thing that one might want to try, thus, is to use E_n 's rather than $K(n)$'s, so that one has generalized character theory ([14]) at hand. Strickland and Turner in [40] used this approach to describe $p^{-1}E_{n*}(CS^0)$ in terms of formal group laws. Here CS^0 is the disjoint union of $B\Sigma_i$'s, related to QS^{2r} 's by Thom isomorphisms and Snaith splitting, whose group completion has the homotopy type of QS^0 . Furthermore, Strickland carried out more detailed analysis of $E_n^*(CS^0)$ using rich structures of CS^0 to give its formal group theoretic description [39]. After the first version of this paper was submitted, Strickland went on to obtain ([38]) a functorial description of

$$\pi_*(L_{K(n)}(E_n \wedge QX))$$

when $\pi_*(L_{K(n)}(E_n \wedge X))$ is the completion of a free module concentrated in even degrees. Here $\pi_*(L_{K(n)}(E_n \wedge -))$ replaces $E_{n*}(-)$.

On the other hand, the result by Kudo and Araki can be reinterpreted in terms of cohomology using a result due to Lannes and Zarati [27]. Let D be the destabilization functor, i.e., the left adjoint to the forgetful functor from the category of unstable algebras over the Steenrod algebra to the category of modules over the Steenrod algebra, and D^i its i^{th} left derived functor. Then their results imply the isomorphism $PH^*(QX; Z/2) \cong \bigoplus_i \Sigma^2 D^i \Sigma^{2-i} \tilde{H}^*(X)$ where P denotes the module of primitives, although they do not state it in this way. This isomorphism can be seen using the relationship between $PH^*(QX; Z/2)$ and $R_n H^*(X; Z/2)$ which seems to be well-known to experts but whose proof does not appear in the literature, where R_n is as defined in [27]. We will discuss it in full detail in a subsequent work [24].

This result suggests that in the case of generalized cohomology theories too, cohomology operations should play a major role. In [23] the author introduced the notion of the destabilization functor for the BP-cohomology. It is not a precise analogue of D above, but an analogue of the composition $U \circ D$ where U is the universal enveloping unstable algebra functor, that is, the left adjoint to the forgetful functor from the category of unstable algebras over the Steenrod algebra to that of unstable modules over the Steenrod algebra. More precisely:

Definition 0.2. Denote by D_{BP} the left adjoint to the forgetful functor from the category of BP-unstable algebras introduced by Boardman, Johnson, and Wilson ([6]), to the category of modules over the Landweber-Novikov algebra, $\text{BP}^*(\text{BP})$.

And the author proved:

THEOREM 0.3 ([23]). *Let X be a (-1) -connected spectrum whose stable cells are concentrated in even degrees and whose BP-cohomology is finitely generated as a $\text{BP}^*(\text{BP})$ -module. Then the natural map $D_{\text{BP}}(\text{BP}^*(X)) \rightarrow \text{BP}^*(\Omega^\infty X)$ is an isomorphism of BP^* -algebras.*

The statement and the proof in [23] contain minor errors; see Section 7. This result notably applies to $\text{BP}^*(QS^{2j})$. We also note that there are many other infinite loop spaces that fit into this category whose mod p ordinary cohomology is still unknown. Similar results concerning K -theory were also proved by Bousfield, namely:

THEOREM 0.4 ([8, Th. 8.3, Cor. 8.6]). *If E is a spectrum with $K^*(E; Z_p^\wedge)$ torsion-free, then $K^*(\Omega_0^\infty E; Z_p^\wedge)$ is naturally isomorphic to $WK^*(E; Z_p^\wedge)_H$. Furthermore, if E is 0-connected and $H^i(E; Z_p^\wedge) = 0$ for $i = 1, 2$, then $K^*(\Omega^\infty E; Z_p^\wedge)$ is naturally isomorphic to $TK^*(E; Z_p^\wedge)$. Here T is the free*

θ^p -ring functor, i.e., the left adjoint to the forgetful functor from the category of θ^p -rings to that of profinite p -groups, $K^*(E; Z_p^\wedge)_H$ is a certain enrichment of $K^*(E; Z_p^\wedge)$, and W is an appropriate adjoint functor that takes into account this enrichment.

We will discuss the relationship between the Bousfield functors T , W and the K -theory version of our destabilization functor in [24]. Furthermore, we know the algebra structure of $K(n)_*(QS^i)$'s:

THEOREM 0.5 ([23]). *$K(n)_*(QS^{2m})$ ($m \geq 0$) is a polynomial algebra concentrated in even degrees. $K(n)_*(QS^0)$ is a tensor product of a polynomial algebra concentrated in even degrees with $K(n)_*[Z]$. $K(n)_*(QS^{2m-1})$ is an exterior algebra with generators in odd degrees.*

It was shown in [25] that these algebras are also cofree as coalgebras. On the other hand, their Hopf algebra structures still remain to be studied.

Now, the purpose of this paper is to generalize the results above. First of all, as we deal with the spaces that are not necessarily finite, we should take into account the topology on their BP-cohomology. Unfortunately abelian topological groups do not form an abelian category, so we need to take care to set up a correct framework to define the destabilization functor.

Definition 0.6. We will call \mathcal{M}_{BP} and \mathcal{K}_{BP} respectively the category of stable BP-cohomology modules defined in [5] and that of unstable BP-cohomology algebras defined in [6] respectively, with the following additional requirements:

- (i) The filtration is over Z .
- (ii) The elements of degree n have filtration at least n .

Now, denote $\mathcal{K}_{0\text{BP}}$ the category of augmented unstable BP-cohomology algebras, that is, the category whose objects are unstable BP-cohomology algebras equipped with the augmentation to the coefficient ring BP^* , and whose morphisms are morphisms of unstable BP-cohomology algebras that respect the augmentation.

The Brown-Peterson cohomology of a spectrum or a pointed space (not necessarily of finite type) is equipped with the skeletal filtration and become objects of \mathcal{M}_{BP} or $\mathcal{K}_{0\text{BP}}$ respectively. (The systematic use of the skeletal filtration causes some inconvenience. However, it allows us to deal with the categorical sums more easily.) Thus \mathcal{M}_{BP} is the algebraic model for the category of spectra, $\mathcal{K}_{0\text{BP}}$ that of pointed spaces, and the augmentation ideal functor \mathcal{I} from $\mathcal{K}_{0\text{BP}}$ to \mathcal{M}_{BP} corresponds to the suspension spectrum functor Σ^∞ . Note also that the cokernel exists in the categories \mathcal{M}_{BP} and $\mathcal{K}_{0\text{BP}}$. It is nothing but the algebraic cokernel equipped with the quotient filtration.

Definition 0.7. A stable BP-cohomology module is called *free* if it is in the essential image of the left adjoint to the forgetful functor to the category of graded sets. A stable BP-cohomology module is called *well-presented* if it is isomorphic in \mathcal{M}_{BP} to the cokernel of a map of the form $F^1 \rightarrow F^0$ where F^i 's are free. We name the full subcategory of \mathcal{M}_{BP} formed by well-presented objects \mathcal{M}'_{BP} . An unstable BP-cohomology algebra is said to be well-presented if it is well-presented as a stable BP-cohomology module. We call the full subcategory of \mathcal{K}'_{0BP} formed by well-presented objects \mathcal{K}'_{0BP} .

The point is that in general, as far as the module structure is concerned, one can express any module as a quotient of a free one, but there is no guarantee that the original topology would coincide with the quotient topology. We avoid this problem by considering only those modules that have the “correct” topology. Note also that if X is a finite-type wedge of suspensions of BP, then $BP^*(X)$ is free. If X is an arbitrary wedge of suspensions of BP, i.e., $X \cong \vee_i \Sigma^{d_i} BP$, then the completion with respect to the skeletal topology of the direct sum $\oplus_i BP^*(\Sigma^{d_i} BP)$ is free, and it is contained in $BP^*(X)$. It is dense if one considers the finite-subcomplex topology defined in Section 7. Now we can introduce our destabilization functor:

PROPOSITION-DEFINITION 0.8. *The augmentation ideal functor from \mathcal{K}'_{0BP} to \mathcal{M}'_{BP} admits a left adjoint, which we call the destabilization functor and we note it \mathcal{D} .*

For free objects in \mathcal{M}'_{BP} that are completions of $\oplus_i BP^*(\Sigma^{d_i} BP)$, one defines the value of \mathcal{D} to be the completions of $\otimes_i BP^*(\underline{BP}_{d_i})$, where $\underline{BP}_{d_i} \cong \Omega^\infty \Sigma^{d_i} BP$. Note that \mathcal{D} is isomorphic to $BP^*(\Pi_i(\underline{BP}_{d_i}))$ if $\Pi_i(\underline{BP}_{d_i})$ (equivalently $\vee_i \Sigma^{d_i} BP$) is of finite type. Otherwise it is dense in $BP^*(\Pi_i(\underline{BP}_{d_i}))$ with respect to the finite-subcomplex topology. Since any object in \mathcal{M}'_{BP} is a quotient of a free one, we can define \mathcal{D} noting that it has to be right exact. Of course, we will need to know if the BP-cohomology of our spaces are well-presented. For this purpose, we introduce yet another definition that is easier to check:

Definition 0.9. Let X be a space or a spectrum. $BP^*(X)$ is said to be *well-generated* if $BP^*(X) \hat{\otimes}_{BP^*} (Z/p) \hookrightarrow H^*(X; Z/p)$.

In most cases when the BP cohomology of a space is known, it is well-generated. However, the result of Inoue [18] seems to imply that $BP^*(BSO(6))$ is not. Anyhow, we will prove the following:

LEMMA 0.10. *For a spectrum or space X , if $BP^*(X)$ is well-generated, then it is well-presented.*

Now we are ready to state the generalization of the preceding results. It is quite natural to limit ourselves to the spaces X such that $\mathrm{BP}^*(X)$ satisfies Landweber's exact-functor-theorem's hypothesis to have Kunnet's isomorphism at hand. Thus our first main result is that the functor Q preserves these properties. That is:

THEOREM 0.11. *Let X be a connected space such that $\mathrm{BP}^*(X)$ is Landweber-flat, i.e., it satisfies one of the equivalent conditions in Theorem 3.6. Then $\mathrm{BP}^*(QX)$ satisfies the same conditions. Furthermore, if $\mathrm{BP}^*(X)$ is well-generated, then so is $\mathrm{BP}^*(QX)$.*

Now that we are assured that our $\mathrm{BP}^*(QX)$ lies in the right category, we can compare it with $\widetilde{\mathcal{DBP}}^*(X)$ and we have:

THEOREM 0.12. *Let X be a connected space, satisfying one of the equivalent conditions in Theorem 3.6, whose BP cohomology is well-generated. Then the natural map $\widetilde{\mathcal{DBP}}^*(X) \rightarrow \mathrm{BP}^*(QX)$ is an isomorphism in $\mathcal{K}'_{0\mathrm{BP}}$.*

We can also generalize Theorem 0.3:

THEOREM 0.13. *Let X be a (-1) -connected spectrum which has stable cells only in even degrees. Then the natural map $\mathcal{DBP}^*(\Sigma^i X) \rightarrow \mathrm{BP}^*(\underline{X}_i)$ is an isomorphism if $i \geq 0$.*

What this means is as follows: Q factors as the composition $\Omega^\infty \Sigma^\infty$ where Σ^∞ is the functor which associates to a space its suspension spectrum, Ω^∞ its right adjoint. On the other hand, BP -cohomology of a spectrum takes value in the category in the modules over $\mathrm{BP}^*(\mathrm{BP})$ whereas BP -cohomology of a space takes value in the category of BP -unstable algebras, and Σ^∞ is compatible with the augmentation ideal functor \mathcal{I} , which means that \mathcal{D} is an algebraic model for Ω^∞ . Thus the composition $\mathcal{D} \circ \mathcal{I}$ can be regarded as an algebraic counterpart of the composition of the functors $X \mapsto \Sigma^\infty X \mapsto QX$. Our theorem shows that it is a good model. This description of \mathcal{D} looks quite abstract. However, it can be described completely algebraically and concretely. As a matter of fact we first obtain an algebraic answer for Morava K -theories and BP -cohomology as follows, then identify our answer with the result of the destabilization.

THEOREM 0.14. *Let X be a space as in Theorem 0.12. Furthermore let $\{f_i : X \rightarrow \underline{\mathrm{BP}}_{d_i} | i \in I\}$ be a set of topological $\mathrm{BP}^*(\mathrm{BP})$ -module generators for $\widetilde{\mathrm{BP}}^*(X)$ (that is, the $\mathrm{BP}^*(\mathrm{BP})$ -submodule generated by f_i 's is dense), $\{g_j : \bigvee_i \Sigma^{d_i} \mathrm{BP} \rightarrow \bigvee_j \Sigma^{e_j} \mathrm{BP}\}$ be a set that generates topologically a complete set of relations (i.e., the exact sequence $0 \leftarrow \mathrm{BP}^*(X) \leftarrow \mathrm{BP}^*(\bigvee_i \Sigma^{d_i} \mathrm{BP}) \leftarrow \mathrm{BP}^*(\bigvee_j \Sigma^{e_j} \mathrm{BP})$ exists). Then there are*

(i) *an exact sequence of Hopf algebras*

$$K(n)_* \rightarrow K(n)_*(QX) \rightarrow K(n)_*(\Pi_i \underline{BP}_{d_i}) \rightarrow K(n)_*(\Pi_j \underline{BP}_{e_j})$$

(ii) *and a coexact sequence of algebras*

$$BP^* \leftarrow BP^*(QX) \leftarrow BP^*(\Pi_i \underline{BP}_{d_i}) \leftarrow BP^*(\Pi_j \underline{BP}_{e_j}).$$

Our description of $K(n)_*(QX)$ and $BP^*(QX)$ in Theorem 0.14 may not look algebraic. However, we will explain how we can reduce everything to a pure algebra (provided that one has the $BP^*(BP)$ -module presentation of $\widetilde{BP}^*(X)$, but this is again a purely algebraic question) using the determination of $E_*(\underline{BP}_*)$ for complex oriented homology theories E by Ravenel and Wilson [33]. As far as the algebra and coalgebra structure is concerned, one can be more explicit (again, the following will be proved before being used in the proof of Theorem 0.14) and generalize Theorem 0.5.

THEOREM 0.15. *Let X be as above. Then $K(n)_*(QX)$ is a free commutative algebra. Furthermore, $K(n)_*(Q\Sigma X)$ is a cofree cocommutative coalgebra.*

Another natural question to ask here is how the set of BP^* -module generators of $BP^*(QX)$ for such spaces (since we know by Theorem 0.11 that they get detected by mod p ordinary cohomology) can be described in terms of $H^*(QX)$. Denote by M_X the image of the Thom map $\rho_X : BP^*(X) \rightarrow H^*(X)$. Then this is equivalent to the knowledge of M_{QX} . Our answer to this question is as follows:

THEOREM 0.16. *Let X be a connected space with the property that $BP^*(X^j) \cong BP^*(X)^{\hat{\otimes} j}$, and $BP^*(X)^{\hat{\otimes}_{BP^*}} Z/p \hookrightarrow H^*(X; Z/p)$. Then $M_{QX} = C$ where C will be defined as in Proposition 5.1.*

In this paper we use the following convention. BP will denote the p -complete version of the Brown-Peterson spectrum (what would normally be denoted as BP_p^\wedge) with the coefficient ring $BP_* \cong Z_p^\wedge[v_1, v_2, \dots]$, $|v_n| = 2(p^n - 1)$, $K(n)$ the usual n^{th} Morava K -theory with $K(n)_* = Z/p[v_n, v_n^{-1}]$, $E(n)$ the Johnson-Wilson theory with $E(n)_* = Z_{(p)}[v_1, \dots, v_n, v_n^{-1}]$, H the mod p ordinary (co)homology. Throughout the main text of the paper, p will be an odd prime. However, most of our results also hold for $p = 2$. Necessary modifications are indicated in the appendix. A “space” will mean a pointed topological space with the homotopy type of a CW-complex of finite type unless otherwise specified. A generalized cohomology of a space is topologized via the skeletal filtration unless otherwise specified, and it is with respect to this topology that we take the completed tensor products. Thus we ignore the

“ p -adic part” of the inverse limit topology on $\mathrm{BP}^*(X) \cong \lim_{n,i} \mathrm{BP}^*(sk_n X)/p^i$. As a matter of fact, this does not make much difference since we take into account the module structure over BP^* , thus over the p -adics, and the finite type hypotheses imply that the skeleton topology together with the module structure over the p -adics suffices to determine the inverse limit topology.

The organization of this paper is as follows. In Section 1 we collect the facts on QX that are necessary for us. In Section 2, we define the notion of Dyer-Lashof length-like filtration, which is used repeatedly in the paper. In Section 3 we review and generalize relevant results in [34]. In Section 4 we use the results in Section 3 and a result by Hunton on the behavior of the Atiyah-Serre-Hirzebruch spectral sequence for a wreath product to show that many properties that $\mathrm{BP}^*(X)$ possesses are passed onto $\mathrm{BP}^*(D_p X)$ and thus to $\mathrm{BP}^*(QX)$. In Section 5 under the assumption that $\mathrm{BP}^*(X) \hat{\otimes}_{\mathrm{BP}^*} (Z/p) \subset H^*(X)$ and that $\mathrm{BP}^*(X)$ satisfies Landweber’s exact-functor-theorem’s hypothesis, we determine the image of the Thom map $\mathrm{BP}^*(QX) \rightarrow H^*(QX)$. In Section 6 we use these results to conclude that $K(n)_*(QX)$ injects to a product of $K(n)_*(\underline{\mathrm{BP}}_i)$ ’s, and deduce from it that $K(n)_*(QX)$ is a free commutative algebra. Then we proceed further to show that the cokernel of the map $K(n)_*(QX) \rightarrow \otimes K(n)_*(\underline{\mathrm{BP}}_i)$ again injects to a product of $K(n)_*(\underline{\mathrm{BP}}_i)$ ’s, and get a completely algebraic description of these objects.

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1. Preliminaries

In this section we collect mostly well-known facts on infinite loop spaces needed later in the paper.

Definition 1.1. Let $I = (\varepsilon_1, s_1, \dots, \varepsilon_k, s_k)$ such that $s_j \geq \varepsilon_j$ and $\varepsilon_j = 0$ or 1. Define the degree (d), the excess (e), the length (l), and the presence of Bockstein at the end (b) of I by

$$\begin{aligned} d(I) &= \sum_{j=1}^k [2(s_j(p-1) - \varepsilon_j)], \\ e(I) &= 2s_1 - \varepsilon_1 - \sum_{j=2}^k [2(s_j(p-1) - \varepsilon_j)], \\ l(I) &= k, \\ b(I) &= \varepsilon_1. \end{aligned}$$

I is said to be *admissible* if $ps_j - \varepsilon_j \geq s_{j-1}$ for $2 \leq j \leq k$.

For any such sequence I (not necessarily admissible), one has a corresponding homology operation on E_∞ spaces $Q^I = \beta^{\varepsilon_1} Q^{s_1} \dots \beta^{\varepsilon_k} Q^{s_k}$, that raises the degree of elements by $d(I)$ and vanishes on elements of degree greater than $e(I)$.

THEOREM 1.2 ([9]). *Let X be a connected space, and Λ a basis for $\tilde{H}_*(X)$. Then $H_*(QX)$ is a free commutative algebra on generators $Q^I(x)$ ($x \in \Lambda$), where I is admissible, $e(I) + b(I) > \deg(x)$. Now, $H_*(CS^0)$ is a free commutative algebra on generators $Q^I([1])$, where I is admissible, $e(I) > 0$. Finally, $H_*(QS^0)$ is the algebra generated by $H_*(CS^0)$, and $[-1]$, subject to the relation $[1] \cdot [-1] = 1$. In the above, $[i]$ denotes the image of the element $i \in \pi_0(QS^0) \cong Z$ or the element $i \in \pi_0(CS^0) \cong Z^+$ by Hurewicz homomorphism $\pi_0(-) \rightarrow H_0(-)$.*

Since the sphere spectrum is a ring spectrum, its multiplication induces a pairing $QS^i \times QS^j \rightarrow QS^{i+j}$. When $i = j = 0$, this pairing agrees with the map induced by $CS^0 \times CS^0 \rightarrow CS^0$ whose components are given by the maps induced by $\Sigma_a \times \Sigma_b \rightarrow \Sigma_{ab}$. This induces a pairing in homology, denoted by \circ : $H_*(CS^0) \otimes H_*(CS^0) \rightarrow H_*(CS^0)$. Since any spectrum, thus in particular a suspension spectrum, is a module spectrum over the sphere spectrum, one gets a pairing $QS^0 \otimes QX \rightarrow QX$, which often is called the composition pairing, as it agrees with the colimit of the maps given by the composition

$$\begin{aligned} \Omega^n S^n \times \Omega^n \Sigma^n X &= \text{Map}_*(S^n, S^n) \times \text{Map}_*(S^n, \Sigma^n X) \\ &\rightarrow \text{Map}_*(S^n, \Sigma^n X) = \Omega^n \Sigma^n X. \end{aligned}$$

We still denote the induced pairing in homology $H_*(QS^0) \otimes H_*(QX) \rightarrow H_*(QX)$ by \circ . Furthermore, the usual Pontrjagin product of $H_*(QX)$ will be denoted by \star or just by juxtaposition. We will need the following.

THEOREM 1.3 (cf. [29]).

- (i) (*May's formula*) $Q^s[1] \circ x = \Sigma_{t \geq 0} Q^{s+t} P_*^t x$
- (ii) (*May's formula*) $\beta Q^s[1] \circ x = \Sigma_{t \geq 0} \beta Q^{s+t} P_*^t x - \Sigma_{t \geq 0} Q^{s+t} P_*^t \beta x$
- (iii) (*Nishida relation*) $P_*^r(Q^s(x)) = \Sigma_i (-1)^{r+i} \binom{(p-1)(s-r)}{r-pi} Q^{s-r+i} P_*^i(x)$
- (iv) (*Nishida relation*)

$$\begin{aligned} P_*^r \beta(Q^s(x)) &= \Sigma_i (-1)^{r+i} \binom{(p-1)(s-r)-1}{r-pi} \beta Q^{s-r+i} P_*^i(x) \\ &\quad + \Sigma_i (-1)^{r+i} \binom{(p-1)(s-r)-1}{r-pi-1} Q^{s-r+i} P_*^i \beta(x) \end{aligned}$$

- (v) $Q^s \sigma(x) = \sigma(Q^s(x))$
- (vi) $Q^s(x) = x^p$ if $s = (p-1)\deg(x)$.

Here P_*^r denotes the dual Steenrod reduced power operation.

Note that the formulae i) and ii) can be combined into a single formula:

$$\beta^\varepsilon Q^s[1] \circ x = \sum_{t \geq 0} \beta^\varepsilon Q^{s+t} P_*^t x - \varepsilon \sum_{t \geq 0} Q^{s+t} P_*^t \beta x.$$

Next we need to know $H^*(B\Sigma_p)$.

PROPOSITION 1.4 (e.g. [37]). $H^*(B\Sigma_p) \hookrightarrow H^*(BZ/p) = \Lambda(x) \otimes Z/p[y]$, the image is the subalgebra generated by y^{p-1} and xy^{p-2} .

We denote by $e_{2i(p-1)}$ the element in $H_*(B\Sigma_p)$ or $H_*(BZ/p)$ that is dual to $y^{i(p-1)}$. Then by definition [2], [9], in $H_*(B\Sigma_p) \subset H_*(CS^0)$, $Q^i[1] = e_{2i(p-1)}$. We note that, since $\beta(x) = y$, an easy Atiyah-Hirzebruch spectral sequence argument shows that the image of $BP^*(B\Sigma_p) \rightarrow H^*(B\Sigma_p)$ is the subalgebra generated by y^{p-1} .

Finally, we will need the following:

Definition 1.5. For a topological space X , $CX = \cup_n E\Sigma_n \times_{\Sigma_n} X^n / \sim$, where \sim is the equivalence defined in [28]. Define a filtration F on CX by

$$F_i(CX) = \cup_{n \leq i} \text{Im}(E\Sigma_n \times_{\Sigma_n} X^n \rightarrow CX).$$

Define DX to be $\vee_i F_i(CX)/F_{i-1}(CX)$.

Remark 1.6. It is well-known and easy to see that $F_i(CX)/F_{i-1}(CX)$ is homeomorphic to $E\Sigma_n^+ \wedge X^n$.

THEOREM 1.7 (e.g. [36], [3]). For a connected space X , QX is stably homotopy equivalent to DX , and CX is homotopy equivalent to QX .

2. Dyer-Lashof length-like filtration

In this section, we introduce the notion of Dyer-Lashof length-like filtration, and discuss its properties.

Definition 2.1. Let A be an algebra augmented over a field k . By $\text{Ind}(A)$ we denote the module of indecomposables, i.e., $\text{Ind}(A) = I/I^2$, where $I = \text{Ker}(A \rightarrow k)$.

We use this notation instead of the traditional Q to avoid confusion with other Q 's used throughout the paper.

Definition 2.2. Fix $d \in \mathbb{Z}$. Denote by $\mathbb{Z}_{\geq d}$ the set of integers greater than or equal to d . Let $\{A_{i,*} | i \in \mathbb{Z}_{\geq d}\}$ be a family of bi-graded Hopf algebras augmented over a field k with characteristic p . Suppose that this family is equipped with a suspension map, i.e., a morphism of k -vector spaces $\sigma : A_{i,*} \rightarrow A_{i+1,*+1}$. We say that the family $\{A_{i,*} | i \in \mathbb{Z}_{\geq d}\}$ together with σ form a $\mathbb{Z}_{\geq d}$ -indexed family of graded algebras with Dyer-Lashof length-like filtration if they are equipped with an increasing filtration F on each $A_{i,*}$, $k = F_0(A) \subset F_1(A) \subset \cdots \subset F_n(A) \subset \cdots \subset A$ satisfying the following properties:

- (i) Each $A_{i,*}$ is a free commutative algebra (in a graded sense).
- (ii) σ factors through $\text{Ind}(A)$, and its image is contained in PA .
- (iii) The decomposition $A_{i,*} \cong P_{i,*} \otimes E_{i,*}$ with P polynomial and E exterior holds as Hopf algebras.
- (iv) σ induces an isomorphism $\text{Ind}P_{i,*} \rightarrow \text{Ind}E_{i+1,*+1}$, an injection $\text{Ind}E_{i,*} \rightarrow P_{i+1,*+1}$.
- (v) F is exhaustive; i.e., $\bigcup_j F_j(A_{i,*}) = A_{i,*}$.
- (vi) $F_l(A_i) \cdot F_k(A_i) \subset F_{l+k}(A_i)$.
- (vii) A_i is isomorphic as an algebra to its associate graded object with respect to the filtration F with the induced multiplication.
- (viii) F is compatible with σ ; i.e., $\sigma(F_j(A_i)) \subset F_j(A_{i+1})$.
- (ix) σ induces isomorphisms

$$F_1(A_{i,*})/F_0(A_{i,*}) \cong \text{colim}(A_{i,*} \xrightarrow{\sigma} A_{i+1,*+1} \xrightarrow{\sigma} \cdots).$$

Remark 2.3. Let A_i be the subalgebra of $H_*(QS^i; \mathbb{Z}/p)$ generated by the elements of the form $Q^I(\iota_i)$ where ι_i is the fundamental class in $H_i(QS^i; \mathbb{Z}/p)$ and where I contains no Bockstein. Then it becomes a \mathbb{Z}^+ -indexed family of algebras with Dyer-Lashof length-like filtration by defining σ to be the restriction of the homology suspension map and F by $F_m(A)$ to be the span of monomials of weight less than or equal to m , where the weight is defined by $\text{weight}(Q^I(\iota)) = p^{l(I)}$, $\text{weight}(xy) = \text{weight}(x) + \text{weight}(y)$. This is the origin of the name of Dyer-Lashof length-like filtration.

The following three propositions are straightforward consequences of the definition.

PROPOSITION 2.4. *Let A, B be $\mathbb{Z}_{\geq d}$ -indexed families of algebras equipped with a Dyer-Lashof length-like filtration. Then so is $A \otimes B$ with the suspension given by*

$$\sigma : A \otimes B \rightarrow \text{Ind}(A \otimes B) \cong \text{Ind}(A) \oplus \text{Ind}(B) \xrightarrow{\sigma \oplus \sigma} A \oplus B \rightarrow A \otimes B$$

and the usual tensor product filtration.

PROPOSITION 2.5. *A direct limit of inclusions compatible with the filtration and the suspension of $Z_{\geq d}$ -indexed families of algebras equipped with a Dyer-Lashof length-like filtration is again a $Z_{\geq d}$ -indexed family of algebras equipped with a Dyer-Lashof length-like filtration.*

PROPOSITION 2.6. *Let A be a $Z_{\geq d}$ -indexed family of algebras equipped with a Dyer-Lashof length-like filtration. If I_i 's form a family of ideals of A_i 's compatible with the suspension such that A_i/I_i 's are free commutative algebras, then A_i/I_i 's form a $Z_{\geq d}$ -indexed family of algebras equipped with a Dyer-Lashof length-like filtration.*

For future use, we record some examples.

PROPOSITION 2.7. *If X satisfies the hypotheses of Theorem 0.12, then $\{K(n)_*(Q\Sigma^r X), r \geq 0\}$ forms a Z^+ -indexed family of algebras equipped with a Dyer-Lashof length-like filtration.*

Proof. Define an increasing filtration F on $K(n)_*(Q\Sigma^m X)$ by

$$F_i(K(n)_*(Q\Sigma^m X)) = \text{Im}(K(n)_*(\cup_{j \leq i} E\Sigma_j \times_{\Sigma_j} X^j) \rightarrow K(n)_*(Q\Sigma^m X)).$$

The assertions (i), (iii), and (iv) will be proved in Section 6. Theorem 1.7 implies (vii), (ii) follows from the property of the homology suspension, and (v), (vi), and (ix) are obvious from the definition. Thus it remains to show (viii).

This seems to be well-known to experts, but does not seem to be in the published literature, so we record a proof here. According to Proposition 5.2 of [28], one has the following commutative diagram where $C_n X$ denotes a certain combinatorial model for $\Omega^n \Sigma^n X$.

$$\begin{array}{ccc} C_n X & \longrightarrow & C_{n+1} X \\ \downarrow & & \downarrow \\ \Omega^n \Sigma^n X & \longrightarrow & \Omega^{n+1} \Sigma^{n+1} X \end{array}$$

Furthermore, the right vertical arrow factors as

$$C_{n+1} X \xrightarrow{\beta_n} \Omega C_n \Sigma^n X \xrightarrow{\gamma_n} \Omega^{n+1} \Sigma^{n+1} X$$

according to [28, Prop. 5.4]. Its proof indicates that β_n 's and γ_n 's are compatible with inclusions $C_n X \hookrightarrow C_{n+1} X$, $\Omega C_n \Sigma X \hookrightarrow \Omega C_{n+1} \Sigma X$, and $\Omega^n \Sigma^n X \hookrightarrow \Omega^{n+1} \Sigma^{n+1} X$, and that $\beta_n(F_l X) \subset \Omega F_l(\Sigma X)$. Therefore one can pass to the colimit to get a commutative diagram

$$\begin{array}{ccc}
CX & \xrightarrow{\text{id}} & CX \\
\downarrow & & \downarrow \searrow \beta \\
QX & \longrightarrow & \Omega Q\Sigma X
\end{array}
\quad \begin{array}{c} \nearrow \\ \Omega C\Sigma X \end{array}$$

such that $\beta(F_l(X)) \subset \Omega(F_l(\Sigma X))$. Now, forgetting the CX on the upper-right corner and by taking the adjoint, one gets the following commutative diagram:

$$\begin{array}{ccc}
\Sigma CX & \xrightarrow{j} & C\Sigma X \\
\downarrow & & \downarrow \\
\Sigma QX & \longrightarrow & Q\Sigma X
\end{array}$$

with $j(\Sigma F_l(X)) \subset F_l(\Sigma X)$. Since the homology suspension map $h_*(QX) \rightarrow h_{*+1}(Q\Sigma X)$ is the composition:

$$h_*(QX) \rightarrow h_{*+1}(\Sigma QX) \xrightarrow{\rho_*} h_*(Q\Sigma X),$$

one gets the desired result. \square

PROPOSITION 2.8. *Let X be a (-1) -connected spectrum whose stable cells are all in even degrees. Then $K(n)_*(\underline{X}_i)$ ($i \in \mathbb{Z}^+$) forms a family of algebras with Dyer-Lashof length-like filtration. In particular, this applies to the case when $X = \text{BP}$.*

Proof. According to [23] there is an isomorphism of algebras $K(n)_*(\underline{X}_i) \cong \otimes K(n)^*(QS^j)$ corresponding to a stable cellular decomposition of X . Although there is nothing canonical in this decomposition, once one fixes the decomposition for $K(n)_*(\underline{X}_0)$, one can choose the rest to be compatible with the suspension homomorphism. Thus we get the desired result using Propositions 2.4 and 2.5. \square

Now we state a very useful property of Dyer-Lashof length-like filtrations.

PROPOSITION 2.9. *Let A be a $\mathbb{Z}_{\geq d}$ -indexed family of algebras equipped with a Dyer-Lashof length-like filtration. Let B be another $\mathbb{Z}_{\geq d}$ -indexed family of algebras equipped with suspension maps satisfying the properties (i)–(iii) in Definition 2.2, such that σ sends the exterior part into the polynomial part and vice versa. Suppose that $f_i : A_i \rightarrow B_i$ is a homomorphism such that f_i 's*

commute with the suspension and they respect the tensor product decomposition of (iii). If

$$\operatorname{colim}_i f_i : \operatorname{colim}(A_{i,*} \xrightarrow{\sigma} A_{i+1,*+1} \xrightarrow{\sigma} \cdots) \rightarrow \operatorname{colim}(B_{i,*} \xrightarrow{\sigma} B_{i+1,*+1} \xrightarrow{\sigma} \cdots)$$

is a monomorphism, then so is $\operatorname{Ind}(f_i) : \operatorname{Ind}(A_i) \rightarrow \operatorname{Ind}(B_i)$ for each i .

Proof. We prove by induction on l that $F_{p^l}(\operatorname{Ind}(A_i))$ injects to $\operatorname{Ind}(B_i)$. Condition (ix) combined with the assumption of the proposition provides the first step. From conditions (ii), (iii) and (iv), we deduce that if

$$x \in (\operatorname{Im} \sigma : \operatorname{Ind}(E_{i,*}) \rightarrow P_{i+1,*+1}) \cap \operatorname{Ker}(P_{i+1,*+1} \rightarrow QP_{i+1,*+1})$$

then there exists y such that $x = y^{p^r}$, and $y \notin \operatorname{Ker}(P_{i+1,*+1} \rightarrow QP_{i+1,*+1})$. Now suppose that $a \neq 0$ is in the kernel of the map $F_{p^l}(\operatorname{Ind}(A_i)) \rightarrow \operatorname{Ind}(B)$. As $\operatorname{colim}_i f_i$ is a monomorphism, there exists $r > 0$ such that $\sigma^r(a) \neq 0$, $\sigma^{r+1}(a) = 0$. Thus the arguments above show that there exists b such that $\sigma^r(a) = b^{p^s}$ and that b reduces nontrivially to $\operatorname{Ind}(A_{i+r})$. By (vii) we can conclude that $b \in F_{p^{l-s}}(A)$. However, by the induction hypothesis, b maps nontrivially to $\operatorname{Ind}(B_i)$. Since b has to be primitive this means that $f_i(b)$ is in the exterior part of B_i , which means that $\sigma(f_i(b))$ is in the polynomial part. However, since $\sigma(b)$ is in the exterior part, $\sigma(f_i(b))$ has to be trivial. This contradicts condition (iv). \square

Remark 2.10. This is essentially how it was shown in [42] that the subalgebra of $H_*(QS^i)$ mentioned in Remark 2.3 injects to $H_*(\underline{\mathbf{BP}}_i)$.

3. How to recover BP-cohomology from Morava K -theory

In this section we recall relevant results in [34] and generalize them to suit our purpose. There exist generalized cohomology theories $E(k, n)$ and $P(n)$ with the ring of coefficients $E(k, n)_* \cong E(n)_*/(p, v_1, \dots, v_{n-1})$, $P(n)_* \cong \mathbf{BP}_*/(p, v_1, \dots, v_{n-1})$.

LEMMA 3.1 ([34]). *If $K(n)^{\operatorname{odd}}(X) = 0$ then $E(k, n)^{\operatorname{odd}}(X) = 0$ for $0 \leq k \leq n$, and $E(k, n)^*(X)$ has no v_k -torsion.*

THEOREM 3.2 ([34]). *Consider the following conditions.*

- (i) $K(n)^{\operatorname{odd}}(X) = 0$ for an infinite number of n .
- (ii) $E(k, n)^{\operatorname{odd}}(X) = 0 \ \forall k, 0 < k < n$ for an infinite number of n .
- (iii) $P(k)^{\operatorname{odd}}(X) = 0$ for all k .

- (iv) $K(k)^{\text{odd}}(X) = 0$ for all k .
- (v) $E(k, n)^*(X)$ is v_k -torsion free for all $0 < k < n$.
- (vi) $P(k)^*(X)$ is v_k -torsion free for all k .
- (vii) (p, v_1, v_2, \dots) is a regular sequence in $\text{BP}^*(X)$.
- (viii) $\text{BP}^*(X) \hat{\otimes}_{\text{BP}^*} P(k)^*$ is isomorphic to $P(k)^*(X)$ for all k .
- (ix) $\text{BP}^*(X) \hat{\otimes}_{\text{BP}^*} K(k)^*$ is isomorphic to $K(k)^*(X)$ for all k .
- (x) $\text{BP}^*(X) \hat{\otimes}_{\text{BP}^*} K(k)^*$ surjects to $K(k)^*(X)$ for all k .
- (xi) $\text{BP}^*(X) \hat{\otimes}_{\text{BP}^*} E(k, n)^*$ surjects to $E(k, n)^*(X)$ for any $n \geq k \geq 0$.

The conditions from (i) to (iv) are equivalent, and they imply the rest.

THEOREM 3.3 ([34]). *Let X_i , $i = 1, 2, 3$ be spaces satisfying one of the conditions from (i) to (iv) of the theorem above. Let $f : X_1 \rightarrow X_2$, $g : X_2 \rightarrow X_3$ be maps with $g \circ f$ null-homotopic. Then,*

- (i) *If $K(n)^*(g)$ is mono for all n then so is $\text{BP}^*(g)$.*
- (ii) *If $K(n)^*(f)$ is epi for all n then so is $\text{BP}^*(f)$.*
- (iii) *Furthermore if all the spaces are H -spaces and maps are H -maps such that*

$$K(n)_*(X_1) \xrightarrow{K(n)_*(f)} K(n)_*(X_2) \xrightarrow{K(n)_*(g)} K(n)_*(X_3)$$

is an exact sequence of Hopf algebras, then

$$\text{BP}^*(X_3) \xrightarrow{\text{BP}^*(g)} \text{BP}^*(X_2) \xrightarrow{\text{BP}^*(f)} \text{BP}_*(X_1)$$

is a coexact sequence of augmented BP^ -algebras. That is, $\text{BP}^*(f)$ is the cokernel of the map $\text{BP}^*(g)$ in the category of augmented BP^* -algebras. More concretely, $\text{BP}^*(g)$ induces an isomorphism between the quotient of $\text{BP}^*(X_2)$ by the image of the augmentation ideal of $\text{BP}^*(X_3)$ by $\text{BP}^*(f)$ and $\text{BP}^*(X_1)$.*

THEOREM 3.4 ([34]). *Let X, Y be spaces satisfying one of the conditions from (i) to (iv). Then*

$$\text{BP}^*(X \times Y) \cong \text{BP}^*(X) \hat{\otimes}_{\text{BP}^*} \text{BP}^*(Y).$$

Remark 3.5. Note that this does not follow from Theorem 3.2 and Landweber's exact functor theorem unless Y is finite. A naive "proof" would involve commuting a direct limit with an inverse limit.

Now we generalize these results.

THEOREM 3.6. *The conditions from (vi) to (x) in Theorem 3.2 are equivalent. They are also equivalent to (v) with $0 < k < n$ replaced by $0 \leq k < n$ (called condition (v)'). Furthermore, it suffices to assume one of these equivalent conditions on spaces appearing in Theorems 3.3 and 3.4 to obtain the same conclusion.*

Proof. It is well-known that condition (vi) implies (vii) that implies (viii) (this can be shown easily, inductively, from the cofibration sequence $P(k) \xrightarrow{v_k} P(k) \rightarrow P(k+1)$). In [34] it was shown that $P(k)^*(X) \hookrightarrow \bigoplus_{n>k} E(k, n)^*(X)$, so that (v)' implies (vi). Since by Morava's little structure theorem ([19], [44, Prop. 1.9] for the present form)

$$P(k)^*(X) \hat{\otimes}_{P(k)^*} (K(k)^*) \cong K(k)^*(X),$$

(viii) implies (ix) which obviously implies (x). The cofibration sequence

$$E(k, n) \xrightarrow{v_k} E(k, n) \rightarrow E(k+1, n)$$

and the fact that the map from BP to $E(k+1, n)$ factors through $E(k, n)$ can be used to show that (xi) implies (v)'. The same cofibration sequence and the fact that the filtration by the power of v_k is complete in $BP^*(X)$ prove that (x) implies (xi) by downward induction on k , where (x) serves as the starting point of the induction. This finishes the proof of the equivalence of the conditions listed. The proof of Theorems 3.3 and 3.4 does not really rely on the properties from (i) to (iv), but only uses the properties (v) and (vi) (and other properties that hold for $BP^*(X)$ for any space X); more precisely the fact that the long exact sequences associated to the aforementioned cofibrations become just a bunch of short exact sequences. Therefore it suffices to assume one of these conditions to get the same conclusion. \square

4. BP-cohomology of the extended power construction

Morava K -theory of the extended power construction was first studied in [15], [14]. The work in [17] treats the most general situation, as well as it deals with the case of other complex oriented cohomology theories including BP-cohomology. We use the results in [17] to obtain:

THEOREM 4.1. *Let X be a connected space satisfying one of the equivalent conditions in Theorem 3.6. Then $D_{Z/p}X$ satisfies the same conditions. Furthermore, if $BP^*(X)$ is well-generated then so is $BP^*(D_{Z/p}X)$. Here $D_{Z/p}X = EZ/p \times_{Z/p} X^p$ where Z/p acts on X^p by permutation.*

Before proving the theorem, we recall a result on the behavior of the Atiyah-Hirzebruch-Serre spectral sequence for the fibration $X^p \rightarrow D_{Z/p}X \rightarrow BZ/p$. First note that, if h is a generalized cohomology theory, the Atiyah-Hirzebruch spectral sequence for the space BZ/p acts on the AHSs in question. We only consider the case when $h^*(X^p)$ is isomorphic to $h^*(X)^{\hat{\otimes} p}$. We say that the AHSs $H^*(BZ/p, h^*(X^p)) \Rightarrow h^*(D_{Z/p}X)$ is simple, if there is no differential in this spectral sequence other than those that are forced by the action of the AHSs $H^*(BZ/p, h^*) \Rightarrow h^*(BZ/p)$.

More precisely,

Definition 4.2 ([17]). If $h^*(X^p) \cong h^*(X)^{\hat{\otimes} p}$, we say that the AHSs $E_2^{*,*} \cong H^*(BZ/p, h^*(X^p)) \Rightarrow E_\infty^{*,*} \cong h^*(D_{Z/p}X)$ is *simple* if $E_2^{0,*} \cong E_\infty^{0,*}$.

We need to know the behavior of this AHSs in more detail. The E_2 term is isomorphic to $A^* \otimes H^*(BZ/p, h^*) \oplus B^*$, where A_* is the span of the elements $\underbrace{a \otimes \cdots \otimes a}_{p \text{ factors}}$, $a \in h^*(X)$, whereas B^* is the span of the elements of the form $\Sigma_{\sigma \in Z/p} \sigma(a_1 \otimes \cdots \otimes a_p)$.

Proof of Theorem 4.1. According to Theorems 2.5 (or the remark preceding it) and 6.1 of [17], the condition for X implies that both the AHSs for $BP^*(D_{Z/p}X)$ and $K(n)^*(D_{Z/p}X)$ are simple. This means for $K(n)$, $E_\infty^{*,*} \cong A^* \otimes H_*(H^*(BZ/p, K(n)_*), v_n Q_n) \oplus B^*$, where A^* and B^* are as above, and Q_n is the n^{th} Milnor's Bockstein operation. Thus as an algebra over $K(n)_*$, it is generated by the elements of A^* , B^* , and the element in $E_\infty^{2,*}$ represented by the element $0 \neq x \in H^2(BZ/p, K(n)_*)$. The collapsing of its BP-counterpart $E'^{*,*}_*$ (which is nothing but the simpleness for BP) implies that all these elements are in the image of the map $E'^{*,*}_\infty \rightarrow E_\infty^{*,*}$ induced by the natural transformation $BP^*(-) \rightarrow K(n)^*(-)$ up to multiplication by some power of v_n . Thus the condition (x) of Theorem 3.6 is easily seen to be satisfied for $D_{Z/p}X$. The second statement follows immediately from the collapsing of the AHSs for $BP^*(D_{Z/p}X)$. \square

As usual, properties that are preserved by the construction $D_{Z/p}$ are preserved by the construction Q . Namely;

Proof of Theorem 0.11. Since these two properties only concern the BP^* -module structure, by Theorem 1.7 it suffices to show these properties for $E\Sigma_n^+ \wedge X^n$. However, one can easily show by transfer arguments that p -locally, these spaces are stable retracts of products of the spaces of the form $D_{Z/p}(\cdots (D_{Z/p}(X)))$ (see, e.g. [30]). Thus one obtains the desired result from Theorem 4.1. \square

5. The image of the Thom map

In this section we describe M_{QX} in terms of M_X with some hypotheses on X . First we establish an upper bound on M_{QX} .

PROPOSITION 5.1. *Let X be a connected space with the property $\mathrm{BP}^*(X^j) \cong \mathrm{BP}^*(X)^{\otimes j}$. Let*

$$B = \{f | f \in H_*(X) = \mathrm{Hom}(H^*(X); Z/p), f \text{ vanishes on } M_X.\}$$

Choose its complement A , i.e., a subspace of $H_(X)$ such that $H_*(X) = A \oplus B$. Then one has*

$$M_{QX} \subset C = \{\phi | H_*(QX) \rightarrow Z/p \text{ such that } \phi(S) = 0\}$$

where S is the ideal generated by the elements of the form $Q^I x$ with $x \in B$ or of the form $Q^J x$ with $x \in A$ and J containing at least one Bockstein, and A and B are considered as subspaces of $H_(QX)$ via the inclusion $H_*(X) \hookrightarrow H_*(QX)$.*

Proof. This will be proved in three steps. First we prove that the elements of the form $Q^I x$ with $x \in B$ or of the form $Q^I x$ with $x \in A$ and I containing at least one Bockstein can be written as a linear combination of elements of the form $Q^{\circ K}[1] \circ z$ with either K containing at least one Bockstein or $z \in B$, where $Q^{\circ K}[1]$ denotes the element $\beta^{\varepsilon_1} Q^{s_1}[1] \circ \dots \circ \beta^{\varepsilon_l} Q^{s_l}[1]$ if $K = \varepsilon_1, s_1, \dots, \varepsilon_l, s_l$. We prove the following two statements by induction on $l(I)$ and $\deg(Q^{I'}(x))$, where if $I = (\varepsilon^1, I^1, \varepsilon^2, I^2, \dots)$ then $I' = (\varepsilon^2, I^2, \dots)$.

(i) Let $N \subset H_*(X)$ be a subspace closed under the action of the Steenrod algebra. Then $Q^I x$ ($x \in N$) can be written as a sum of the elements of the form $Q^{\circ K}[1] \circ z$ with $z \in N$.

(ii) Furthermore suppose that $\beta(H_*(X)) \subset N$. If I contains a Bockstein, then $Q^I x$ ($x \in H_*(X)$) can be written as a sum of the elements of the form $Q^{\circ K}[1] \circ z$ with either K containing a Bockstein or $z \in N$.

To prove the first statement, using May's formula, one gets

$$Q^I(x) = \beta^{\varepsilon^1} Q^{I^1}[1] \circ Q^{I'}(x) - \sum_{t>0} \beta^{\varepsilon^1} Q^{I^1+t} P_*^t Q^{I'}(x) + \varepsilon^1 \sum_{t \geq 0} Q^{I^1+t} P_*^t \beta Q^{I'}(x).$$

Since $l(I') = l(I) - 1$, the first term can be taken care of by induction on l . Using Nishida relations one can rewrite $P_*^t Q^{I'}(x)$ ($t > 0$) and $P_*^t \beta Q^{I'}(x)$ as a linear combination of elements of the form $Q^J(z)$ with $l(J) = l(I')$, z belonging to the orbit of x by the action of the Steenrod algebra, thus belonging to N and $\deg(Q^J(z)) < \deg(Q^{I'}(x))$. Since the sequences (ε^1, I_1, J) have the same length as I , and the degree of $Q^J(z)$ is less than that of $Q^{I'}(x)$, the two summations can be taken care of by the induction hypothesis, which finishes the proof of (i). To prove (ii), when $\varepsilon^1 = 0$, I' still contains a Bockstein, and

the rest of the argument is similar. When $\varepsilon_1 = 1$, one can treat the terms in the two summations similarly, and, to take care of the term $\beta^{\varepsilon_1} Q^{I^1}[1] \circ Q^{I'}(x)$, one applies the case $N = H_*(X)$ of (i) to $Q^{I'}(x)$. Thus one has proved (i) and (ii).

Now note that since the operations P^j 's are covered by some Landweber-Novikov operations, B is stable under the action of the P_*^j 's, and since the Bockstein vanishes on M_X , in $H_*(X)$ the image of β is contained in B . Thus one can take N in the statements above to be B to obtain the desired result.

In the next step, we show that there exists a family of spaces $Y_{k,l,i} \cong (B\Sigma_p)^l \times X^k$ and a map $g_{k,l,i} : Y_{k,l,i} \rightarrow QX$ such that any element of S can be written as a linear combination of elements of the form $g_{k,l,i} * (e_{j_1} \otimes \cdots \otimes e_{j_l} \otimes x_1 \otimes \cdots \otimes x_k) \in H_*(QX)$ where either at least one of j_t 's is congruent to $(-1) \bmod 2(p-1)$ or at least one of the x_k 's is in B .

As a matter of fact it is enough to take the family

$$\begin{aligned} & \left\{ \overbrace{(B\Sigma_p \times \cdots \times B\Sigma_p)}^{m_1 \text{ factors}} \times X \times \cdots \times \overbrace{(B\Sigma_p \times \cdots \times B\Sigma_p)}^{m_k \text{ factors}} \times X \right. \\ & \rightarrow B\Sigma_p^{m_1} \times X \times \cdots \times B\Sigma_p^{m_k} \times X \\ & \rightarrow B\Sigma_{p^{m_1} + \cdots + p^{m_k}} \times X^k \\ & \rightarrow QS^0 \times QX \\ & \left. \rightarrow QX \right\} \end{aligned}$$

where the first map is induced by the multiplication map $B\Sigma_a \times B\Sigma_b \rightarrow B\Sigma_{ab}$, the second by the addition $B\Sigma_a \times B\Sigma_b \rightarrow B\Sigma_{a+b}$, the third by obvious ones, and the last by the composition pairing. Now by definitions $\alpha_1 \otimes \cdots \otimes \alpha_k$ each $\alpha_s \in H_*(B\Sigma_p^{m_s} \times X)$ is mapped to $\mu_*(\alpha_1) \star \cdots \star \mu_*(\alpha_k)$ where μ_* is given by

$$\mu_*((e_{2j_1(p-1)+\varepsilon_1} \otimes \cdots \otimes e_{2j_m(p-1)+\varepsilon_m}) \otimes x) = (\beta^{\varepsilon_1} Q^{j_1}[1] \circ \cdots \circ \beta^{\varepsilon_m} Q^{j_m}[1]) \circ x.$$

Thus one deduces the desired result from the previous step.

Now the proof of the proposition can be completed as follows. The conclusion is equivalent to the vanishing of the restriction to $M_{QX} \otimes S$ of the Kronecker pairing $H^*(X) \otimes H_*(X) \rightarrow Z/p$. However, the assumption on X implies that $BP^*(Y_{k,l,i}) \cong BP^*(B\Sigma_p)^{\hat{\otimes} l} \hat{\otimes}_{BP^*} BP^*(X)^{\hat{\otimes} k}$. Since we know that if $c \in M_{B\Sigma_p}$ then $c(e_j) = 0$ if j is congruent to $(-1) \bmod 2(p-1)$, we see that if $c \in M_{Y_{k,l,i}}$, c vanishes on elements of the form $e_{j_1} \otimes \cdots \otimes e_{j_l} \otimes x_1 \otimes \cdots \otimes x_k$ where either at least one of j_t 's is congruent to $(-1) \bmod 2(p-1)$ or at least one of the x_k 's is in B . Thus if $f \in M_{QX}$, then $g_{k,l,i}^*(f)$ vanishes on such elements. The result of the previous step now implies the desired result. \square

Next we go on to establish a lower bound for M_{QX} .

LEMMA 5.2. *Let $\{f_i : X \rightarrow \underline{\mathbf{BP}}_{d_i} | i \in I\}$ be a set of topological $\mathbf{BP}^*(\mathbf{BP})$ -module generators for $\widehat{\mathbf{BP}}^*(X)$. Then one has*

$$\mathrm{Im}(\otimes_{i \in I} H^*(\Omega^\infty f_i) : H^*(\prod_{i \in I} \underline{\mathbf{BP}}_{d_i}) \rightarrow H^*(QX)) = C$$

where C is as in Proposition 5.1.

Proof. Consider a Z^+ -indexed family of graded algebras $\{H_*(Q\Sigma^n X) | n \in Z^+\}$. We fix a direct sum decomposition as in Proposition 5.1: $H_*(X) = A_X \oplus B_X$ and $H_*(\Sigma^n X) = A_{\Sigma^n X} \oplus B_{\Sigma^n X}$ compatible with the suspension isomorphism. Let $T_{\Sigma^n X}$ denote the subalgebra of $H_*(Q\Sigma^n X)$ generated by the elements of the form $Q_J x$ with $x \in A_{\Sigma^n X}$ and J containing no Bockstein. Then $H_*(Q\Sigma^n X) = T_{\Sigma^n X} \oplus S_{\Sigma^n X}$, where S is as defined in Proposition 5.1. It is easy to see that $\{T_{\Sigma^n X} | n \in Z^+\}$ forms a Z^+ -indexed family of graded algebras with Dyer-Lashof length-like filtration. Furthermore, one has

$$\mathrm{colim}(\rightarrow T_{\Sigma^n X} \rightarrow T_{\Sigma^{n+1} X} \rightarrow \cdots) \cong A.$$

On the other hand, by choice of the maps f_i , we know that $\mathbf{BP}^*(\bigvee_{i \in I} \Sigma^{d_i} \mathbf{BP})$ surjects to $\widehat{\mathbf{BP}}^*(X)$. Thus $H^*(\bigvee_{i \in I} \Sigma^{d_i} \mathbf{BP})$ surjects to M_X (note that the Thom homomorphism $\mathbf{BP}^*(\bigvee_{i \in I} \Sigma^{d_i} \mathbf{BP}) \rightarrow H^*(\bigvee_{i \in I} \Sigma^{d_i} \mathbf{BP})$ is also surjective so that $H^*(\bigvee_{i \in I} \Sigma^{d_i} \mathbf{BP})$ maps to M_X). As the restriction of the pairing $H^*(X)$ and $H_*(X)$ identifies A with the dual of M , A injects to $H_*(\bigvee_{i \in I} \Sigma^{d_i} \mathbf{BP})$. Thus by Theorem 2.9 we see that T_X injects to $H_*(\prod_{i \in I} \underline{\mathbf{BP}}_{d_i})$. On the other hand, since $\mathbf{BP}^*(\underline{\mathbf{BP}}_{d_i})$ surjects to $H^*(\underline{\mathbf{BP}}_{d_i})$, B_X is seen to be in the kernel of $H_*(\Omega^\infty f_i)$. Furthermore, since Bocksteins act trivially on $H_*(\underline{\mathbf{BP}}_{d_i})$ we see that $S_X \subset (\mathrm{Ker} \oplus_i H_*(\Omega^\infty f_i))$. As one has seen $H_*(QX) = S_X \oplus T_X$, this shows that $S_X = \mathrm{Ker}(\oplus_i H_*(\Omega^\infty f_i))$. By dualizing one gets the desired result. \square

Proof of Theorem 0.16. Since $\mathbf{BP}^*(\underline{\mathbf{BP}}_{d_i})$ surjects to $H^*(\underline{\mathbf{BP}}_{d_i})$, Lemma 5.2 implies that $C \subset M_{QX}$. Combining this with Proposition 5.1, one gets the desired result. \square

6. $K(n)_*(QX)$ and $\mathbf{BP}^*(QX)$

In this section we determine $K(n)_*(QX)$ and $\mathbf{BP}^*(QX)$ for the spaces X satisfying the hypotheses of Theorem 0.12. First we prove:

THEOREM 6.1. *Let X be a connected space satisfying one of the equivalent conditions of Theorem 3.6, and*

$$\mathbf{BP}^*(X) \hat{\otimes}_{\mathbf{BP}^*} \mathbb{Z}/p \hookrightarrow H^*(X; \mathbb{Z}/p).$$

Let $\{f_i : X \rightarrow \underline{\mathbf{BP}}_{d_i} \mid i \in I\}$ be a set of $\mathbf{BP}^*(\mathbf{BP})$ -module generators for $\widetilde{\mathbf{BP}}^*(X)$. Then

- (i) $\hat{\otimes}_i \mathbf{BP}^*(f_i)$ is surjective,
- (ii) $\otimes_i K(n)_*(f_i)$ is injective.

Proof. According to Theorem 3.6, one has $\mathbf{BP}^*(X^j) \cong \mathbf{BP}^*(X)^{\hat{\otimes} j}$. Thus we see that M_{QX} agrees with the image of the composition

$$\mathbf{BP}^*(\Pi_{i \in I} \underline{\mathbf{BP}}_{d_i}) \rightarrow \mathbf{BP}^*(QX) \rightarrow H^*(QX).$$

However, we also know from Theorem 0.11 that

$$\mathbf{BP}^*(QX) \hat{\otimes}_{\mathbf{BP}^*} (Z/p) \subset H^*(QX).$$

This concludes the proof of (i). Using Theorem 0.11 one can deduce (ii) from (i). \square

Note that in the above, one can take all d_i to be positive. This easily follows from the fact that $\mathbf{BP}^*(X)$ is well-generated. One can also deduce it by the theorem of Quillen [32] which says that the BP cohomology of a space is always generated by nonnegative degree elements. Thus $K(n)_*(\Pi_{i \in I} \underline{\mathbf{BP}}_{d_i})$ is a free commutative algebra, and by Theorem B.7 of [7] any of its Hopf subalgebras is a free commutative algebra. Thus we obtain the first statement of Theorem 0.15. Now we will go on to show the second statement.

Proof of Theorem 0.15. We will study the bar spectral sequence for the fibration $QX \rightarrow \text{pt} \rightarrow Q\Sigma X$. We have just seen that there is a Hopf algebra isomorphism $K(n)_* QX \cong P_X \otimes E_X$, where P_X is a polynomial algebra concentrated in even degrees and E_X is an exterior algebra generated by odd degree elements. Thus we have

$$\begin{aligned} E^2 &\cong \text{Tor}^{K(n)_*(QX)}(K(n)_*, K(n)_*) \\ &\cong \Gamma(\sigma \text{Ind}(E_X)) \otimes \Lambda(\sigma \text{Ind}(P_X)). \end{aligned}$$

Now, let $J = \{i \in I \mid d_i \text{ is even}\}$, and let Y be the cofiber of the map $f' = \Pi_{i \in J} f_i : X \rightarrow \vee_{i \in J} \Sigma^{d_i} \mathbf{BP}$. Consider the bar spectral sequence associated to the fibration $QX \rightarrow \Pi_{i \in J} \underline{\mathbf{BP}}_{d_i} \rightarrow \Omega^\infty Y$. As the map $K(n)_*(\Pi_{i \in I} \Omega(f_i))$ is injective, we see that the map $K(n)_*(\Pi_{i \in J} \Omega(f_i))$ is injective on the polynomial part. Thus the E^2 term has the form

$$\text{Tor}^{E_X}(K(n)_*, K(n)_*) \otimes K(n)_*(\Pi_{i \in J} \underline{\mathbf{BP}}_{d_i}) // P_X,$$

which is concentrated in even degrees. Therefore it collapses. Next we compare it with the bar spectral sequence above, and we see that the factor $\Gamma(\sigma \text{Ind}(E_X))$ is composed of permanent cycles only. As the other factor $\Lambda(\sigma \text{Ind}(P_X))$ is also

composed of permanent cycles since it is generated by homological degree 1 elements, we see that this spectral sequence collapses as well. Thus $E^2 = E^\infty$ and the E^∞ term is a cofree coalgebra; thus there can be no coalgebra extension and $K(n)_*(Q\Sigma X)$ is a cofree coalgebra. \square

Note that Proposition 5.1 above immediately implies the properties i) and iii) in Definition 2.2. The remaining property iv) follows easily from the collapse of the bar spectral sequence for the fibration $QX \rightarrow \text{pt} \rightarrow Q\Sigma X$ which has just been proved. This completes the proof of Proposition 2.7. We also note some special cases of Proposition 5.1 above.

COROLLARY 1. *Let X be a connected space with $\widetilde{K(n)}_{\text{even}}(X) = 0$ for all n . Then $K(n)_*(QX)$ is an exterior Hopf algebra.*

Proof. Since $K(n)_{\text{odd}}(\Sigma X) = 0$ for all n , by Theorem 3.2 one sees that $\text{BP}^{\text{odd}}(\Sigma X) = 0$; i.e., $\widetilde{\text{BP}}^{\text{even}}(X) = 0$. Thus there is an inclusion of Hopf algebras $K(n)_*(QX) \rightarrow K(n)_*(\Pi_i \underline{\text{BP}}_{d_i})$ where all d_i 's are odd. Since $K(n)_*(\underline{\text{BP}}_{\text{odd}})$ is an exterior Hopf algebra, we get the desired result. \square

COROLLARY 2. *Let X be a connected space with $K(n)_{\text{odd}}(X) = 0$ for all n . Then $K(n)_*(QX)$ is a polynomial algebra.*

Proof. The same arguments as above work except that now all d_i 's are even. \square

For a fixed value of n , fewer f_i 's will suffice in Theorem 6.1, namely:

PROPOSITION 6.2. *Let $\{f_i : X \rightarrow \underline{\text{BP}}_{d_i} \mid i \in I\}$ be maps with the property that $\widetilde{K(n)}_*(X) \xrightarrow{\oplus K(n)_*(f_i)} \oplus \Sigma^{d_i} K(n)_*(\text{BP})$ is injective. Then the map*

$$\text{Ind}(\otimes_i K(n)_*(f_i)) : \text{Ind}(K(n)_*(QX)) \rightarrow \text{Ind}(K(n)_*(\Pi_i \underline{\text{BP}}_{d_i}))$$

is injective.

Proof. Now that one has seen that $K(n)_*(Q\Sigma^r X)$'s are free, the Proposition 2.7 implies that it is equipped with the Dyer-Lashof length-like filtration. Thus we get the result by applying Proposition 2.9. \square

We note another variant which should be of independent interest, namely:

COROLLARY 3. *Let $\{f_i : X \rightarrow \underline{E(n)}_{d_i} \mid i \in I\}$ be maps with the property that $\widetilde{K(n)}_*(X) \xrightarrow{\oplus K(n)_*(f_i)} \oplus \Sigma^{d_i} K(n)_*(E(n))$ is injective. Then the map*

$$\text{Ind}(\otimes_i K(n)_*(f_i)) : \text{Ind}(K(n)_*(QX)) \rightarrow \text{Ind}(K(n)_*(\Pi_i \underline{E(n)}_{d_i}))$$

is injective.

Proof. This follows from the fact that $K(n)_*(\underline{E(n)}_i)$ is a polynomial algebra if i is even and an exterior algebra if i is odd [16], [13]. \square

Notably, if we take X to be a sphere, we can use the unit map for the spectrum $E(n)$. This generalizes the well-known result on injections $K(1)_*(QS^0) \hookrightarrow K(1)_*(BU \times Z)$ and $K(1)_*(QS^2) \hookrightarrow K(1)_*(BU)$ ([12]). (Strictly speaking, the case for QS^0 is not covered by the fact that S^0 is not connected, though it is not difficult to extend our result to this case, which is left as an exercise for interested readers.)

Regard f_i 's as maps of spectra $\Sigma^\infty X \rightarrow \Sigma^{d_i} \text{BP}$. Let C_f denote the cofiber of the map $f = \vee f_i : X \rightarrow \vee \Sigma^{d_i} \text{BP}$. We will now consider $K(n)_*(\Omega^\infty C_f)$.

LEMMA 6.3. *There is a short exact sequence of Hopf algebras*

$$K(n)_*(QX) \rightarrow K(n)_*(\Pi_i \underline{\text{BP}}_{d_i}) \rightarrow K(n)_*(\Omega^\infty C_f).$$

Proof. We consider the bar spectral sequence associated to the fibration $QX \rightarrow \Pi_i \underline{\text{BP}}_{d_i} \rightarrow \Omega^\infty C_f$. By Proposition 6.2 and Theorem 10.8 of [8], we see that the E_2 term $\text{Tor}^{K(n)_*(QX)}(K(n)_*, K(n)_*(\Pi_i \underline{\text{BP}}_{d_i}))$ is concentrated in homological degree zero and isomorphic to $K(n)_*(\Pi_i \underline{\text{BP}}_{d_i}) \otimes_{K(n)_*(QX)} K(n)_*$ so that the SS collapses and we get the desired result. \square

Remark 6.4. Although we can prove the injection $\text{Ind}(K(n)_*(QX)) \hookrightarrow \text{Ind}(K(n)_*(\Pi_i \underline{\text{BP}}_{d_i}))$, as we are dealing with Hopf algebras with periodic gradings, it does not suffice to conclude that $K(n)_*(\Omega^\infty C_f)$ is free.

PROPOSITION 6.5. *$K(n)_*(\Omega^\infty C_f)$ is a free commutative algebra.*

Proof. We use the notation and definitions in the proof of Theorem 0.15. The short exact sequence above splits as the tensor product of the short exact sequences $E_X \rightarrow K(n)_*(\Pi_{i \in I-J} \underline{\text{BP}}_{d_i}) \rightarrow E_C$, $P_X \rightarrow K(n)_*(\Pi_{i \in J} \underline{\text{BP}}_{d_i}) \rightarrow P_C$ with $E_C \otimes P_C \cong K(n)_*(\Omega^\infty C_f)$. Obviously, E_C is an exterior algebra generated by odd degree elements, and P_C is concentrated in even degrees. Thus it suffices to show that P_C is a polynomial algebra. However, we have seen in the proof of Theorem 0.15 that the bar spectral sequence for the fibration $QX \rightarrow \Pi_{i \in J} \underline{\text{BP}}_{d_i} \rightarrow \Omega^\infty Y$ collapses, which implies that P_C injects to $K(n)_*(\Omega^\infty Y)$. Thus it suffices to prove that $K(n)_*(\Omega^\infty Y)$ is a polynomial algebra.

Consider the Eilenberg-Moore spectral sequences (see [35]) for the following three fibrations: $QX \rightarrow \text{pt} \rightarrow Q\Sigma X$, $\Pi_{i \in J} \underline{\text{BP}}_{d_i} \rightarrow \text{pt} \rightarrow \Pi_{i \in J} \underline{\text{BP}}_{d_i+1}$, and $\Omega^\infty Y \rightarrow Q\Sigma X \rightarrow \Pi_{i \in J} \underline{\text{BP}}_{d_i+1}$. We show that they collapse at E_2 and actually converge. We will call the E_2 (thus E_∞) term $E_2(1), E_2(2), E_2(3)$ respectively. For the first fibration, we have seen that $K(n)_*(Q\Sigma X) \cong P_{\Sigma X} \otimes E_{\Sigma X}$, where $P_{\Sigma X}$ is isomorphic to $\Gamma(\sigma(\text{Ind}(E_X)))$ as coalgebras, and

$E_{\Sigma X} \cong \Lambda(\sigma(\text{Ind}(P_X)))$. Thus the E_2 term is

$$\begin{aligned} E_2(1) &\cong \text{Cotor}_{\Gamma(\sigma(\text{Ind}(E_X))) \otimes \Lambda(\sigma(\text{Ind}(P_X)))}(K(n)_*, K(n)_*) \\ &\cong E_X \otimes P_X \\ &\cong K(n)_*(QX) \end{aligned}$$

thus it collapses at E_2 and converges. (Tamaki also constructed an Eilenberg-Moore type spectral sequence that is strongly convergent [41].) Similarly the second collapses and converges. As to the third one, we see that

$$\begin{aligned} E_2(3) &\cong \text{Cotor}_{K(n)_*(\Pi_{i \in J} \underline{\text{BP}}_{d_i+1})}(K(n)_*(Q\Sigma X), K(n)_*) \\ &\cong \text{Cotor}_{K(n)_*(\Pi_{i \in J} \underline{\text{BP}}_{d_i+1})}(P_{\Sigma X} \otimes E_{\Sigma X}, K(n)_*) \\ &\cong P_{\Sigma X} \otimes \text{Cotor}_{K(n)_*(\Pi_{i \in J} \underline{\text{BP}}_{d_i+1})//E_{\Sigma X}}(K(n)_*, K(n)_*) \\ &\cong P_{\Sigma X} \otimes \text{Sym}(\sigma^{-1} \text{Ind}(K(n)_*(\Pi_{i \in J} \underline{\text{BP}}_{d_i+1})//E_{\Sigma X})). \end{aligned}$$

Thus the E_2 term is concentrated in even degrees, and the spectral sequence collapses. Now we need to show its convergence. Note that the computations with the bar spectral sequences in the proof of Theorem 0.15 show that we have the following exact sequence of Hopf algebras:

$$E_X \otimes P_X \rightarrow K(n)_*(\Pi_{i \in J} \underline{\text{BP}}_{d_i}) \rightarrow K(n)_*(\Omega^\infty Y) \rightarrow P_{\Sigma X} \rightarrow K(n)_*.$$

On the other hand, we also have the following exact sequence from above:

$$E_2(1) \rightarrow E_2(2) \rightarrow E_2(3) \rightarrow P_{\Sigma X} \rightarrow K(n)_*.$$

Thus $E_2(3)$ is isomorphic to an associated graded object of $K(n)_*(\Omega^\infty Y)$ so that the spectral sequence converges. Furthermore, as $E_2(3)$ is a polynomial algebra, there can be no nontrivial algebra extension, which shows that $K(n)_*(\Omega^\infty Y)$ is a polynomial algebra as desired. \square

Now we embed $K(n)_*(\Omega^\infty C_f)$ into a more familiar object.

PROPOSITION 6.6. *Let $\{g_i : C_f \rightarrow \underline{\text{BP}}_{e_i} | i \in J\}$ be a set with the property that $K(n)_*(\Pi_i g_i) : K(n)_*(\vee \Sigma^{e_i} \text{BP}) \rightarrow K(n)_*(C_f)$ is a monomorphism, for example a set of topological $\text{BP}^*(\text{BP})$ -module generators for $\text{BP}^*(C_f)$. Then $K(n)_*(\Pi_i \Omega^\infty g_i) : K(n)_*(\Omega^\infty C_f) \rightarrow K(n)_*(\Pi_i \underline{\text{BP}}_{e_i})$ is a monomorphism.*

Proof. By Propositions 2.6 and 6.5, we see that $K(n)_*(\Omega^\infty \Sigma^r C_f)$ ($r \in \mathbb{Z}^+$) forms a \mathbb{Z}^+ -indexed family of free algebras with Dyer-Lashof length-like filtration. Thus one has the desired result. \square

Thus we are ready to identify $K(n)_*(QX)$ as well as $\text{BP}^*(QX)$.

Proof of Theorem 0.14. The first statement is obtained by combining Propositions 6.3 and 6.6. The second follows from the first by Theorems 3.6 and 0.11. The readers may object that $\Pi_j \underline{BP}_{e_j}$ is not necessarily of finite type. However, since this space is torsion-free, one can prove directly the properties from (v)' to (xi) of Theorem 3.2. Thus we can apply Theorem 0.11 to the sequence $QX \rightarrow \Pi_i \underline{BP}_{d_i} \rightarrow \Pi_j \underline{BP}_{e_j}$. \square

Remark 6.7. (i) Of course, to obtain part (i) of Theorem 3.2 for a fixed n , it suffices to assume that one has an exact sequence of the form $0 \rightarrow \widehat{K(n)}_*(X) \rightarrow K(n)_*(\bigvee_i \Sigma^{d_i} BP) \rightarrow K(n)_*(\bigvee_j \Sigma^{e_j} BP)$.

(ii) The computation of $E_*(\underline{BP}_*)$ for complex oriented cohomology theories E in [33] makes the evaluation of the map at the right end of the exact sequence a completely algebraic process, as explained in [6]. It can also be reduced to evaluating g_j 's as cohomology operations on $BP^*(CP^\infty \times \cdots \times CP^\infty)$. This can be seen as follows. According to [21], $E_*(\underline{BP}_{2*})$ is spanned by elements of the form $E_*(\alpha)(\beta)$ where $\alpha \in BP^{2*}(CP^\infty \times \cdots \times CP^\infty) \cong [CP^\infty \times \cdots \times CP^\infty, \underline{BP}_{2*}]$, $\beta \in E_*(CP^\infty \times \cdots \times CP^\infty)$. But $E_*(g_j)(E_*(\alpha)(\beta)) = E_*(g_j(\alpha))(\beta)$, where $g_j(\alpha)$ denotes g_j evaluated on α .

7. The destabilization functor for BP-cohomology

In this section we prove Lemma 0.10 and complete the proof of Theorem 0.12. For this purpose, we start by studying the nature of the skeleton filtration on $BP^*(X)$ when it is well-generated. First we improve Lemma 4.3 of [34].

LEMMA 7.1. *Let X be a spectrum. Denote by $E_{*,*}^*(X)$ and $E_{*,*}^*(sk_m X)$ respectively the Atiyah-Hirzebruch spectral sequences $H^*(X, BP^*) \Rightarrow BP^*(X)$ and $H^*(sk_m X, BP^*) \Rightarrow BP^*(BP_m X)$. Then the natural map of the spectral sequences $E_{s,t}^r(X) \rightarrow E_{s,t}^r(sk_m X)$ induced by the inclusion of the skeleton is an isomorphism for $s \leq m - r + 1$ and a monomorphism for $m - r + 2 \leq s \leq m$.*

Proof. We proceed by induction on r . The assertions are clearly true when $r = 2$. Now suppose that they are true for r . Consider the following commutative diagram.

$$\begin{array}{ccccc}
 E_{s-r,*}^r(X) & \xrightarrow{d_r} & E_{s,*}^r(X) & \xrightarrow{d_r} & E_{s+r,*}^r(X) \\
 \downarrow & & \downarrow & & \downarrow \\
 E_{s-r,*}^r(sk_m X) & \xrightarrow{d_r} & E_{s,*}^r(sk_m X) & \xrightarrow{d_r} & E_{s+r,*}^r(sk_m X)
 \end{array}$$

When $s \leq m - r$, by the induction hypothesis, the right vertical arrow is a monomorphism, and the other two vertical arrows are isomorphisms. Thus after taking the homology in the middle, one sees that the induced map $E_{s,t}^{r+1}(X) \rightarrow E_{s,t}^{r+1}(\mathrm{sk}_m X)$ is an isomorphism. When $m - r + 1 \leq s \leq m$, the induction hypothesis implies that the left vertical arrow is an isomorphism, and the middle one is a monomorphism. Therefore, by passing to the homology, we see that the map $E_{s,t}^{r+1}(X) \rightarrow E_{s,t}^{r+1}(\mathrm{sk}_m X)$ is a monomorphism. Thus we conclude that the assertions hold for any r . \square

LEMMA 7.2. *Let X be a space or spectrum whose BP cohomology is well-generated. Then an element α of $\mathrm{BP}^*(X)$ lies in the kernel of the map $\mathrm{BP}^*(X) \rightarrow \mathrm{BP}^*(\mathrm{sk}_n X)$ if and only if there exist elements $a_i \in \mathrm{BP}^*$, $x_i \in \mathrm{BP}^{l_i}(X)$ such that $l_i > n$ for all i and $\alpha = \sum_i a_i x_i$.*

Proof. The “if” part is obvious. So it suffices to prove the “only if” part. Let $\{e_i | i \in \Lambda\}$ be a set of elements of $\mathrm{BP}^*(X)$ such that $\{\rho_X(e_i) | i \in \Lambda\}$ is a basis for M_X . Since $\mathrm{BP}^*(X)$ is well-generated, $\{e_i | i \in \Lambda\}$ generates $\mathrm{BP}^*(X)$. Thus if α is an element of $\mathrm{BP}^*(X)$, it can be written as

$$\alpha = \sum_i \beta_i e_i$$

with $\beta_i \in \mathrm{BP}^*$. This implies that if it is in $\mathrm{Ker}(\mathrm{BP}^*(X) \rightarrow \mathrm{BP}^*(\mathrm{sk}_n X))$, then

$$\alpha' = \sum_{d_i \leq n} \beta_i e_i \in \mathrm{Ker}(\mathrm{BP}^*(X) \rightarrow \mathrm{BP}^*(\mathrm{sk}_n X))$$

since the other terms are obviously in the kernel. Let $d = \min\{d_i | \beta_i \neq 0\}$. Thus in the Atiyah-Hirzebruch spectral sequence for $\mathrm{BP}^*(\mathrm{sk}_n X)$ we see that $\sum_{d_i=d} \beta_i \otimes e_i$ is a boundary element and in E'_∞ we have the equality $\sum_{d_i=d} \beta_i \otimes e_i = 0$ and we have a nontrivial additive extension of the form

$$\sum_{d_i=d} \beta_i e_i = -\sum_{d < d_i \leq n} \beta_i e_i.$$

On the other hand, as we have seen previously, the map $E_{s,t}^\infty(X) \rightarrow E_{s,t}^\infty(\mathrm{sk}_n X)$ is injective for $s \leq n$. Thus by the naturalness of the Atiyah-Hirzebruch spectral sequence, one can conclude that in $E_{s,t}^\infty(X)$ there is the same type of extension problem, and

$$\sum_{d_i=d} \beta_i e_i = -\sum_{d < d_i \leq n} \beta_i e_i + \sum_{d_i > n} \gamma_i e_i.$$

Thus we have $\alpha = \sum_{d_i > n} \gamma_i e_i + \sum_{d_i > n} \beta_i e_i$ as desired. \square

As a straightforward consequence, we have the following:

PROPOSITION 7.3. *Let X, Y be spectra or spaces and $f : X \rightarrow Y$ a map such that $\mathrm{BP}^*(X)$ and $\mathrm{BP}^*(Y)$ are well-generated and such that $\mathrm{BP}^*(f)$ is surjective. Then the skeletal filtration on $\mathrm{BP}^*(X)$ agrees with the quotient filtration induced from the skeletal filtration on $\mathrm{BP}^*(Y)$.*

Proof of Lemma 0.10. The case of a space follows from the case of a spectrum. When X is a spectrum, it suffices to consider a family of generators $X \xrightarrow{\vee_i f_i} \vee_i \Sigma^{d_i} \mathbf{BP}$, and apply the lemma above. \square

Now we are ready to complete the proof of Theorems 0.12 and 0.13. As we need to deal with a topology on $\mathbf{BP}^*(-)$ that is different from the skeletal topology, we recall:

Definition 7.4. The finite-subcomplex topology on $\mathbf{BP}^*(X)$ (whether X is a space or a spectrum) is the topology in which the system of neighbourhoods of 0 is the set of $\text{Ker}(\mathbf{BP}^*(X) \rightarrow \mathbf{BP}^*(X_\alpha))$ where X_α runs through all finite subcomplexes of X .

Remark 7.5. This topology is often called pro-finite topology in the literature (e.g. [1], [5]). We prefer to rename it because the term pro-finite topology means something else for algebraists (the topology in which the neighbourhoods of 0 are the subgroups of finite index), and it is not too absurd to consider the pro-finite topology in this sense here.

Proof of Theorem 0.12. According to Theorem 0.14, we have the exact sequence of $\mathbf{BP}^*(\mathbf{BP})$ -modules

$$0 \leftarrow \widetilde{\mathbf{BP}}^*(X) \leftarrow \mathbf{BP}^*(Y_1) \leftarrow \mathbf{BP}^*(Y_2)$$

and a coexact sequence of augmented \mathbf{BP}^* -algebras

$$\mathbf{BP}^* \leftarrow \mathbf{BP}^*(QX) \leftarrow \mathbf{BP}^*(\Omega^\infty Y_1) \leftarrow \mathbf{BP}^*(\Omega^\infty Y_2),$$

where Y_1 and Y_2 are wedges of suspensions of \mathbf{BP} . On the other hand, X being of finite type, one can take Y_1 to be of finite type as well, which forces $\Omega^\infty Y_1$ to be of finite type also. Thus one sees that $\mathbf{BP}^*(Y_1)$ is free and $\mathbf{BP}^*(\Omega^\infty Y_1)$ can be identified with $\mathcal{D}(\mathbf{BP}^*(Y_1))$. Unfortunately Y_2 is not necessarily of finite type. If $Y_2 \cong \vee_j \Sigma^{e_j} \mathbf{BP}$, $\mathbf{BP}^*(Y_2)$ is the completion with respect to the finite subcomplex topology of $\bigoplus_j \Sigma^{e_j} \mathbf{BP}^*(\mathbf{BP})$. It contains the completion with respect to the skeletal topology of $\bigoplus_j \Sigma^{e_j} \mathbf{BP}^*(\mathbf{BP})$, which we call M . Thus M is free, dense in $\mathbf{BP}^*(Y_2)$ with respect to the finite subcomplex topology, and complete with respect to the skeletal topology. Since these two topologies are natural, $\text{Im}(M \rightarrow \mathbf{BP}^*(Y_1))$ is dense in $\text{Im}(\mathbf{BP}^*(Y_2) \rightarrow \mathbf{BP}^*(Y_1))$ with respect to the finite subcomplex topology, and complete with respect to the skeletal topology. But these two topologies agree on $\mathbf{BP}^*(Y_1)$, which implies that $\text{Im}(M \rightarrow \mathbf{BP}^*(Y_1))$ coincides with $\text{Im}(\mathbf{BP}^*(Y_2) \rightarrow \mathbf{BP}^*(Y_1))$. Thus we can replace the first exact sequence by the following one:

$$0 \leftarrow \widetilde{\mathbf{BP}}^*(X) \leftarrow \mathbf{BP}^*(Y_1) \leftarrow M.$$

Here the two terms on the right are free. On the other hand, Lemma 0.10 shows that the skeletal filtration on $\widetilde{\mathbf{BP}}^*(X)$ agrees with the quotient filtration

induced from that of $\mathrm{BP}^*(Y_1)$. Thus this is really an exact sequence in $\mathcal{M}'_{\mathrm{BP}}$. Using the definition of \mathcal{D} we can identify $\mathrm{BP}^*(\Omega^\infty Y_1)$ with $\mathcal{D}\mathrm{BP}^*(Y_1)$. Furthermore, by arguments similar to the one above, we can replace $\mathrm{BP}^*(\Omega^\infty Y_2)$ by $\mathcal{D}(M)$ in the coexact sequence above. Again Proposition 7.3 shows that $\mathrm{BP}^*(QX)$ has the quotient filtration and that the sequence is coexact in $\mathcal{K}'_{0\mathrm{BP}}$. Since \mathcal{D} has to be right exact, we obtain the desired result. \square

We can also generalize the result in [23].

Proof of Theorem 0.13. The case in which i is even is essentially already treated in [23, Cor. to Th. 7.3]. However, there is one mistake in the proof in [23]. It was implicitly assumed there that BP-cohomology of a wedge of suspensions of BP is a free module over $\mathrm{BP}^*(\mathrm{BP})$, which is false unless we take into account the topology (thus replacing D defined there by \mathcal{D} defined here) or if we assume that $\mathrm{BP}^*(X)$ is finitely generated as a $\mathrm{BP}^*(\mathrm{BP})$ -module so that we do not have to worry about the topology.

Now let $i = 2j - 1$. Then as was shown in [25] one has the inclusion $\mathrm{Ind}K(n)_*(\underline{X}_{2j-1}) \subset K(n)_*(\underline{X}_{2j})$. Since $PK(n)_*(\underline{X}_{2j-1}) \cong \mathrm{Ind}K(n)_*(\underline{X}_{2j-1})$, in cohomology we have that the map $K(n)^*(\underline{X}_{2j}) \rightarrow \mathrm{Ind}K(n)^*(\underline{X}_{2j-1})$ is an epimorphism. Since $\mathrm{BP}^*(\underline{X}_{2j}) \hat{\otimes}_{\mathrm{BP}^*} K(n)^*$ surjects to $K(n)^*(\underline{X}_{2j})$, one sees that $\mathrm{BP}^*(\underline{X}_{2j-1}) \hat{\otimes}_{\mathrm{BP}^*} K(n)^*$ surjects to $K(n)^*(\underline{X}_{2j-1})$. However, the arguments in [23] show that we have all the exact sequences of Hopf algebras needed in Morava K -theories, so that Theorem 3.6 implies the desired result. \square

Appendix. Modifications for prime 2

In this appendix we treat the case $p = 2$. First of all, we don't have to worry about the possible noncommutativity of Morava K -theory, since in cohomology, all spaces dealt with satisfy $\mathrm{BP}^*(X) \hat{\otimes}_{\mathrm{BP}^*} K(n) \cong K(n)^*(X)$, so that the cup product is commutative. In homology, all H -spaces dealt with have H -maps from each one to another space whose Morava K -homology is known to be commutative; these maps induce monomorphism in Morava K -homology, so that the Morava K -homology of these H -spaces is commutative. There remain two sources of problems. First of all, the square of odd degree elements in commutative graded $\mathbb{Z}/2$ -algebras is not necessarily zero, which requires us to revise the content of Sections 2 and 6. Another thing is that the Adem relations, May's formula, and Nishida relations do not exactly look the way they do when p is odd, which makes us modify the arguments in Section 5 a little bit. Now we list what changes.

In Section 1, we first replace Definition 1.1.

Definition A.1. $I = (s_1, \dots, s_k)$ is called admissible if for $s_j \leq 2s_{j+1}$, the excess, the degree, and the length of I are defined by $d(I) = \sum_{j=1}^k s_j$, $l(I) = k$ and $e(I) = s_1 - \sum_{j=2}^k s_j$.

With this modification, Theorem 1.2 holds as stated, except that now the relevant reference is [2]. The formulae in Theorem 1.3 which do not involve β hold by replacing P^i with Sq^i . In particular we have $\beta Q^{2s} = Q^{2s-1}$. Observe also that the formulae in Theorem 1.3 which involve β hold as well by replacing Q^i with Q^{2i} and P^i with Sq^{2i} . Finally Proposition 1.4 is true modulo the algebra extension $x^2 = y$.

In Section 2, everything remains valid if we replace “free commutative algebra” with “tensor product of a polynomial algebra concentrated in even degrees and an exterior algebra generated by odd degree elements”. The results in this section now can be used in Section 6. However, we need a variant of Proposition 2.9 that can be used in the proof of Lemma 5.2. For this purpose, we change conditions (i) and (iv) of the Definition 2.2 as follows:

(i') Each $A_{i,*}$ is a polynomial algebra, and there exists ε , such that $A_{i,*}$ is generated by even degree elements if and only if i is congruent to $\varepsilon \bmod 2$, and such that $A_{i,*}$ is generated by odd degree elements otherwise.

(iv') σ induces an isomorphism $\text{Ind}(A_{2i+\varepsilon}) \rightarrow \text{Ind}(A_{2i+\varepsilon+1})$ and a monomorphism $\text{Ind}(A_{2i+\varepsilon-1}) \rightarrow A_{2i+\varepsilon}$.

Now with this definition for the algebras with Dyer-Lashof length-like filtration, a variant of Proposition 2.9 holds by requiring B to satisfy the following conditions:

- (i) Each $B_{2i+\varepsilon}$ is a polynomial algebra concentrated in even degrees.
- (ii) Each $B_{2i+\varepsilon-1}$ is an exterior algebra generated by odd degree elements.

The proof is similar to the odd prime case.

In Section 5, the following modifications are required. Wherever we consider an operation Q^I containing at least one Bockstein for each odd prime, we consider an operation Q^I , $I = (s_1, \dots, s_l)$, with at least one s_j being odd. Then the proof of Proposition 5.1 can be proved in a similar way as in the odd prime case, taking into account the observation we made after the modifications on Theorem 1.3. Finally we prove a weakened version of Lemma 5.2. First we show:

LEMMA A.2. *Let $\{f_i : X \rightarrow \underline{\mathbf{BP}}_{d_i} \mid i \in I\}$ be a set that reduces to a $Z/2$ -basis for $M_X \subset \tilde{H}^*(X)$. Then one has*

$$\text{Im}(\otimes_{i \in I} H^*(\Omega^\infty f_i) : H^*(\Pi_{i \in I} \underline{\mathbf{BP}}_{d_i}) \rightarrow H^*(QX)) = C$$

where C is as in Proposition 5.1.

Proof. We proceed as in the odd prime case to take a subalgebra $T_{\Sigma^n X}$ of $H_*(Q\Sigma^n X)$ similarly. If we give an increasing filtration on $T_{\Sigma^n X}$ by defining $F_j(T_{\Sigma^n X})$ to be the subalgebra generated by the image of Dyer-Lashof operations on the elements in $H_j(\Sigma^n X)$, and that on $H_*(\Pi_{i \in I} \underline{BP}_{d_i+n})$ by defining

$$F_j(\Pi_{i \in I} \underline{BP}_{d_i+n}) \cong H_j(\Pi_{i \in I, d_i+n \leq j} \underline{BP}_{d_i+n}),$$

then $H_*(\Pi_i f_i)$ respects this filtration. However, F_l/F_{l-1} of the former is just $(A_{\Sigma^n X})_l \otimes T_{\Sigma^n S^{l-n}}$ whereas that of the latter is $(A_{\Sigma^n X})_l \otimes H_*(\underline{BP}_l)$. We know that the former injects to the latter either by our modified version of Proposition 2.9 or by [42]. Thus we see that T_X injects to $H_*(\Pi_{i \in I} \underline{BP}_{d_i})$. The rest of the proof does not require modification. \square

Fewer f_i 's will suffice, at least when X satisfies the hypotheses of Theorem 0.12. We will come back to this point later.

In Section 6, we first have to weaken Theorem 6.1. Namely we have to take $\{f_i : X \rightarrow \underline{BP}_{d_i} | i \in I\}$ be a set that reduces to a $Z/2$ -basis for $M_X \subset \tilde{H}^*(X)$. Then using Lemma A.2 instead of Lemma 5.2, one can prove the theorem in a similar way. Next, throughout the section, “polynomial algebra” should be replaced with “polynomial algebra concentrated in even degrees” and “exterior algebra” should be replaced with “exterior algebra generated by odd degree elements”. Then the rest of the section becomes true. Note that using Proposition 6.2, one sees that Theorem 6.1 holds without changing the family $\{f_i : X \rightarrow \underline{BP}_{d_i} | i \in I\}$ under the hypotheses of Theorem 0.12. Thus under these assumptions, one can prove Lemma 5.2 for the original family $\{f_i : X \rightarrow \underline{BP}_{d_i} | i \in I\}$ in the statement.

Throughout Section 6, “free commutative algebra” should be replaced with “polynomial algebra concentrated in even degrees tensored with exterior algebra generated by odd degree elements”, and “cofree cocommutative coalgebra” with “divided power coalgebra concentrated in even degrees tensored with exterior coalgebra generated by odd degree elements”. Some of the spectral sequences may have possible nontrivial algebra extension problems due to the fact that the squares of odd degree elements are not automatically zero. However, using the naturality arguments and comparison with appropriate spectral sequences, one can always show that the statements in this section remain true after the modification mentioned above.

All the rest of the article remains true as stated.

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REFERENCES

- [1] J. F. ADAMS, A variant of E. H. Brown's representability theorem, *Topology* **10** (1971), 185–198.

- [2] S. ARAKI and T. KUDO, Topology of H_n -spaces and H -squaring operations, *Mem. Fac. Sci. Kyusyu Univ. Ser. A* (1956), 85–120.
- [3] M. G. BARRATT and P. J. ECCLES, Γ^+ -structures-III. The stable structure of $\Omega^\infty \Sigma^\infty A$, *Topology* **13** (1974), 199–207.
- [4] M. BENDERSKY, E. B. CURTIS, and H. R. MILLER, The unstable Adams spectral sequence for generalized homology, *Topology* **17** (1978), 229–248.
- [5] J. M. BOARDMAN, Stable operations in generalized cohomology, in *The Handbook of Algebraic Topology*, North-Holland, Amsterdam, 585–686, 1995.
- [6] J. M. BOARDMAN, D. C. JOHNSON, and W. S. WILSON, Unstable operations in generalized cohomology, *The Handbook of Algebraic Topology*, 687–828, North-Holland, Amsterdam, 1995.
- [7] A. K. BOUSFIELD, On p -adic λ -rings and the K -theory of H -spaces, *Math. Z.* **223** (1996), 483–519.
- [8] ———, On λ -rings and the K -theory of infinite loop spaces, *K-Theory* **10** (1996), 1–30.
- [9] E. DYER and R. K. LASHOF, Homology of iterated loop spaces, *Amer. J. Math.* **84** (1962), 35–88.
- [10] L. HODGKIN, The K -theory of some well-known spaces. I. QS^0 , *Topology* **11** (1972), 371–375.
- [11] ———, Dyer-Lashof operations in K -theory, in *New Developments in Topology (Proc. Sympos. Algebraic Topology, Oxford, 1972)*, 27–32, *London Math. Soc. Lecture Note Ser.*, No. 11, Cambridge Univ. Press, London, 1974.
- [12] L. HODGKIN and V. P. A. SNAITH, The K -theory of some more well-known spaces, *Illinois J. Math.* **22** (1978), 270–278.
- [13] M. J. HOPKINS and J. R. HUNTON, On the structure of spaces representing a Landweber exact cohomology theory, *Topology* **34** (1995), 29–36.
- [14] M. J. HOPKINS, N. J. KUHN, and D. C. RAVENEL, Generalized group characters and complex oriented cohomology theories, *J.A.M.S.* **13** (2000), 553–594.
- [15] J. R. HUNTON, Morava K -theory of wreath products, *Math. Proc. Cambridge Philos. Soc.* **107** (1990), 309–318.
- [16] ———, Detruncating Morava K -theory, in *Adams Memorial Symposium on Algebraic Topology* **2** (Manchester, 1990), *London Math. Society Lect. Note Ser.* **176**, Cambridge Univ. Press, London, 1992.
- [17] ———, The complex oriented cohomology of extended powers, *Ann. Inst. Fourier* **48** (1998), 517–534.
- [18] K. INOUE, The Brown-Peterson cohomology of $BSO(6)$, *J. Math. Kyoto Univ.* **32** (1992), 655–666.
- [19] D. C. JOHNSON and W. S. WILSON, BP operations and Morava’s extraordinary K -theories, *Math. Z.* **144** (1975), 55–75.
- [20] T. KASHIWABARA, Mod p K -theory of $\Omega^\infty \Sigma^\infty X$ revisited, *Math. Proc. Cambridge Philos. Soc.* **114** (1993), 219–221.
- [21] ———, Hopf rings and unstable operations, *J. Pure Appl. Algebra* **94** (1994), 183–193.
- [22] ———, $K(2)$ -homology of some infinite loop spaces, *Math. Z.* **218** (1995), 503–518.
- [23] ———, Brown-Peterson cohomology of $\Omega^\infty \Sigma^\infty S^{2n}$, *Quart. J. Math.* **49** (1998), 345–362.
- [24] ———, Unstable Hopf algebras, destabilization and cohomology of infinite loop spaces, in preparation.
- [25] T. KASHIWABARA, N. P. STRICKLAND, and P. R. TURNER, The Morava K -theory Hopf ring for BP , in *Algebraic Topology: New Trends in Localization and Periodicity*, *Progr. Math.* **136**, Birkhäuser, Basel, 1996.
- [26] T. KASHIWABARA and W. S. WILSON, The Morava K -theory and Brown-Peterson cohomology of spaces related BP , preprint.
- [27] J. LANNES and S. ZARATI, Sur les foncteurs dérivés de la déstabilisation, *Math. Z.* **194** (1987), 25–59.

- [28] J. P. MAY, *The Geometry of Iterated Loop Spaces*, *Lect. Notes in Math.* **271**, Springer-Verlag, New York, 1972.
- [29] ———, The homology of E_∞ spaces, in *The Homology of Iterated Loop Spaces*, *Lect. Notes in Math.* **533**, 1–68, Springer-Verlag, New York, 1976.
- [30] J. E. MCCLURE, In mod p K -theory of QX , in *H_∞ Ring Spectra and their Applications*, *Lect. Notes in Math.* **1176**, 291–383, Springer-Verlag, New York, 1986.
- [31] H. R. MILLER and V. P. A. SNAITH, On $K_*(Q\mathbf{R}P^n; P\mathbf{Z}/2)$, in *Current Trends in Algebraic Topology, Part I*, *CMS Conf. Proc.* **2** (1982), 233–243.
- [32] D. QUILLEN, Elementary proofs of some results of cobordism theory using Steenrod operations, *Adv. in Math.* **7** (1971), 29–56.
- [33] D. C. RAVENEL and W. S. WILSON, The Hopf ring for complex cobordism, *J. Pure Appl. Algebra* **9** (1977), 241–280.
- [34] D. C. RAVENEL, W. S. WILSON, and N. YAGITA, Brown-Peterson cohomology from Morava K -theory, *K-Theory* **15** (1998), 147–199.
- [35] L. SMITH, On the construction of the Eilenberg-Moore spectral sequence, *Bull. A. M. S.* **75** (1969), 873–878.
- [36] V. P. A. SNAITH, A stable decomposition of $\Omega^n S^n X$, *J. London Math. Soc.* **7** (1974), 577–583.
- [37] N. E. STEENROD, *Cohomology Operations*, *Ann. of Math. Studies*, No. 50, Princeton Univ. Press, Princeton, NJ, 1962.
- [38] N. P. STRICKLAND, unpublished.
- [39] ———, Morava E -theory of symmetric groups, *Topology* **37** (1998), 757–779; Correction: *ibid.* **38** (1999), 931.
- [40] N. P. STRICKLAND and P. R. TURNER, Rational Morava E -theory of DS^0 , *Topology* **36** (1997), 137–151.
- [41] D. TAMAKI, A dual Rothenberg-Steenrod spectral sequence, *Topology* **33** (1994), 631–662.
- [42] W. S. WILSON, The Ω -spectrum for Brown-Peterson cohomology I, *Comment. Math. Helv.* **48** (1973), 45–55; corrigendum: *ibid.* **48** (1993), 194.
- [43] ———, *Brown-Peterson Homology: An Introduction and Sampler*, *CBMS Reg. Conf. Ser. Math.* **48**, Conf. Board of Math. Sciences, Washington, D.C., 1982.
- [44] ———, Brown-Peterson cohomology from Morava K -theory II, *K-Theory* **17** (1999), 95–101.

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